Vortex rigid motion in quasi-geostrophic shallow-water equations

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Abstract

In this paper, we prove the existence of analytic relative equilibria with holes for quasi-geostrophic shallow-water equations. More precisely, using bifurcation techniques, we establish for any $\mathbf{m}$ large enough the existence of two branches of $\mathbf{m}$-fold doubly-connected V-states bifurcating from any annulus of arbitrary size.

Contents

1 QGSW equations and main result

In this work, we are concerned with the quasi-geostrophic shallow-water equations with a parameter $\lambda \geqslant 0$, which is a two dimensional active scalar equation taking the form

$$(\text{QGSW})_\lambda \begin{cases} \partial_t \boldsymbol{q} + \mathbf{v}\cdot\nabla \boldsymbol{q} = 0, & (t,x)\in\mathbb{R}_+\times\mathbb{R}^2 \\ \mathbf{v} = \nabla^\perp(\Delta-\lambda^2)^{-1}\boldsymbol{q}, & \\ \boldsymbol{q}(0,\cdot) = \boldsymbol{q}_0, & \end{cases} \qquad \text{where} \quad \nabla^\perp = \begin{pmatrix} -\partial_2 \\ \partial_1 \end{pmatrix}. \tag{1.1}$$

The involved quantities are the divergence-free velocity field $\mathbf{v}$ and the potential vorticity $\boldsymbol{q}$ which is a scalar function. The parameter λ stands for the inverse Rossby radius defined in the literature by

$$\lambda = \frac{\omega_c}{\sqrt{gH}},$$

where g is the gravity constant, H is the mean active layer depth and ω_c is the Coriolis frequency, assumed to be constant. Notice that the case $\lambda = 0$ corresponds to the velocity-vorticity formulation of Euler equations. The system (1.1) is commonly used to track the dynamics of the atmospheric and oceanic circulation at large scale motion. For a general review about the asymptotic derivation of the these equations from the rotating shallow water equations we refer for instance to [39, p. 220].

The main purpose of this paper is to explore the emergence of time periodic solutions in the patch form close to the annulus of radii 1 and b for the system $(\text{QGSW})_\lambda$ with fixed $\lambda > 0$ and $b \in (0,1)$. Recall that a vortex patch means a solution of (1.1) with initial condition being the characteristic function of a bounded domain

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$D_0 \subset \mathbb{R}^2$, that is $\boldsymbol{q}_0 = \mathbf{1}_{D_0}$. Actually, this structure is conserved in time due to the transport structure of (1.1), and one gets

$$\boldsymbol{q}(t,\cdot) = \mathbf{1}_{D_t} \quad \text{where} \quad D_t := \boldsymbol{\Phi}_t(D_0),$$

with $\boldsymbol{\Phi}_t : \mathbb{R}^2 \to \mathbb{R}^2$ is the flow map associated to $\mathbf{v}$, defined through the ODE

$$\partial_t \boldsymbol{\Phi}_t(z) = \mathbf{v}(t, \boldsymbol{\Phi}_t(z)) \quad \text{and} \quad \boldsymbol{\Phi}_0 = \mathrm{Id}_{\mathbb{R}^2}. \tag{1.2}$$

In this framework, of bounded datum with compact support, existence and uniqueness follow in a standard way from Yudovich approach implemented for Euler equations and which can be adapted here in a similar way. These structures can be considered as a toy model to simulate hurricanes motion in the context of geophysical flows. In the smooth boundary case, their dynamics is completely described by the evolution of the interfaces surrounding the patch according to the contour dynamics equation given by

$$\Big[\partial_t \gamma(t,\theta) - \mathbf{v}\big(t, \gamma(t,\theta)\big)\Big] \cdot \mathbf{n}\big(t, \gamma(t,\theta)\big) = 0, \tag{1.3}$$

where $\gamma(t,\cdot) : \mathbb{T} \to \partial D_t$ is a C^1 parametrization of the boundary of the patch and $\mathbf{n}(t,\cdot)$ is an outward normal vector to the boundary. We may refer to [28, 29] for a detailed derivation of this equation for active scalar models. We are particularly interested in the existence of ordered structures moving without shape deformation, called *V-states*. More precisely, we shall focus on the existence of uniformly rotating vortex patches about their center of mass, that can be fixed at the origin, and with a constant angular velocity $\Omega \in \mathbb{R}$, namely

$$\boldsymbol{q}(t,\cdot) = \mathbf{1}_{D_t} \quad \text{with} \quad D_t = e^{\mathrm{i}t\Omega} D_0. \tag{1.4}$$

In the present work we explore the case of doubly-connected V-states with $\mathbf{m}$-fold symmetry. To fix the terminology, a bounded open domain D_0 is said doubly-connected if

$$D_0 = D_1 \backslash \overline{D_2},$$

where D_1 and D_2 are two bounded open simply-connected domains with $\overline{D_2} \subset D_1$. This means that the boundary of D_0 is given by two interfaces, one of them is contained in the open region delimited by the second one. According to the structure of $(QGSW)_\lambda$, every radial initial domain D_0 generates a trivial stationary solution, and therefore a V-state rotating with any angular velocity. Basic examples are given by the discs in the simply-connected case or the annuli in the doubly-connected case. The first non-trivial examples of uniformly rotating solutions for Euler equations are Kirchhoff ellipses which rotate with the angular velocity $\Omega = \frac{ab}{(a+b)^2}$ where a and b are the semi-axes of the ellipse (see [34] and [4, p. 304]). In [8] Deem and Zabusky established numerically the existence of simply-connected rotating patches with $\mathbf{m}$-fold symmetry for $\mathbf{m} > 2$. An analytical proof based on bifurcation theory and complex analysis tools was performed by Burbea in [5] showing the existence of $\mathbf{m}$-fold (for any $\mathbf{m} \in \mathbb{N}^*$) symmetric V-states bifurcating from Rankine vortices with angular velocity $\Omega_{\mathbf{m}} := \frac{\mathbf{m}-1}{2\mathbf{m}}$. In the spirit of Burbea's work, a lot of results on $\mathbf{m}$-fold V-states have been obtained both for simply and doubly-connected cases for Euler, $(SQG)_\alpha$ and $(QGSW)_\lambda$ equations in the past decade. We may refer to [6, 9, 16, 17, 19, 20, 24, 38]. From this long list, we shall make some comments on two contributions from [9, 24] related to the current work. In [24, Thm. B], the authors proved for Euler equations that under the condition

$$1 + b^{\mathbf{m}} - \frac{\mathbf{m}(1-b^2)}{2} < 0,$$

one can find two branches of $\mathbf{m}$-fold doubly-connected V-states bifurcating from the normalized annulus A_b, defined by

$$A_b := \big\{ z \in \mathbb{C} \quad \text{s.t.} \quad b < |z| < 1 \big\} \quad \text{for} \quad b \in (0,1) \tag{1.5}$$

at the following angular velocities

$$\Omega^{\pm}_{\mathbf{m}}(b) = \frac{1-b^2}{4} \pm \frac{1}{2\mathbf{m}} \sqrt{\left(\frac{\mathbf{m}(1-b^2)}{2} - 1\right)^2 - b^{2\mathbf{m}}}. \tag{1.6}$$

Burbea's result has been extended for $(QGSW)_\lambda$ in [9, Thm. 5.1], where it is shown the existence of branches of $\mathbf{m}$-fold symmetric V-states ($\mathbf{m} \geq 2$) bifurcating from Rankine vortex $\mathbf{1}_{\mathbb{D}}$, with $\mathbb{D}$ being the unit disc, at the angular velocitity

$$\Omega_{\mathbf{m}}(\lambda) = I_1(\lambda)K_1(\lambda) - I_{\mathbf{m}}(\lambda)K_{\mathbf{m}}(\lambda) \tag{1.7}$$

where I_m and K_m are the modified Bessel functions of first and second kind, respectively. We may refer to the Appendix A for the definitions and some basic properties of these functions. We also notice that more analytical and numerical experiments were carefully explored in [9, 10] dealing in particular with the imperfect bifurcation and the response of the bifurcation diagram with respect to the parameter λ.

We emphasize that different studies around this subject have been recently investigated by several authors, we refer for instance to [12, 13, 15, 18, 22, 27] and the references therein.
The main contribution of this paper is to establish for $(QGSW)_\lambda$ the existence of branches of bifurcation in the doubly-connected case, generalizing the result of [24]. More precisely, we prove the following result.

Theorem 1.1. *Let $\lambda > 0$ and $b \in (0,1)$. There exists $N(\lambda,b) \in \mathbb{N}^*$ such that for every $\mathbf{m} \in \mathbb{N}^*$, with $\mathbf{m} \geqslant N(\lambda,b)$, there exist two curves of $\mathbf{m}$-fold doubly-connected V-states bifurcating from the annulus A_b defined in* (1.5)*, at the angular velocities*

$$\begin{aligned}\Omega_{\mathbf{m}}^{\pm}(\lambda,b) = &\frac{1-b^2}{2b}\Lambda_1(\lambda,b) + \frac{1}{2}\Big(\Omega_{\mathbf{m}}(\lambda) - \Omega_{\mathbf{m}}(\lambda b)\Big)\\ &\pm \frac{1}{2b}\sqrt{\Big(b\big[\Omega_{\mathbf{m}}(\lambda) + \Omega_{\mathbf{m}}(\lambda b)\big] - (1+b^2)\Lambda_1(\lambda,b)\Big)^2 - 4b^2\Lambda_{\mathbf{m}}^2(\lambda,b)},\end{aligned}$$

where $\Omega_{\mathbf{m}}(\lambda)$ is defined in (1.7) *and*

$$\Lambda_m(\lambda,b) := I_m(\lambda b)K_m(\lambda),$$

with I_m and K_m being the modified Bessel functions of first and second kind. In addition, the boundary of each V-state is analytic.

Before sketching the proof some remarks are in order.

Remark 1.1. *The spectrum is continuous with respect to λ and b. In particular, when we shrink $\lambda \to 0$ we find the spectrum of Euler equations detailed in* (1.6)*. However, when we shrink $b \to 0$ we obtain in part the simply connected spectrum* (1.7) *. In other words,*

$$\begin{cases} \Omega_{\mathbf{m}}^{\pm}(\lambda,b) \underset{\lambda\to 0}{\longrightarrow} \Omega_{\mathbf{m}}^{\pm}(b)\\ \Omega_{\mathbf{m}}^{+}(\lambda,b) \underset{b\to 0}{\longrightarrow} \Omega_{\mathbf{m}}(\lambda).\end{cases}$$

These asymptotics are obtained for sufficiently large values of $\mathbf{m}$. For more details see Lemma 3.2.

Now, we intend to discuss the key steps of the proof of Theorem 1.1. The following notation will be used throughout the paper.

- We denote by $\mathbb{D}$ the unit disc. Its boundary, the unit circle, is denoted by $\mathbb{T}$.
- Let $f : \mathbb{T} \to \mathbb{C}$ be a continuous function. We shall use the following notation throughout the paper

$$\fint_{\mathbb{T}} f(\tau)d\tau := \frac{1}{2\mathrm{i}\pi}\int_{\mathbb{T}} f(\tau)d\tau := \frac{1}{2\pi}\int_0^{2\pi} f\left(e^{\mathrm{i}\theta}\right)e^{\mathrm{i}\theta}d\theta,$$

  where $d\tau$ stands for the complex integration.

First, in Section 2, we reformulate the vortex patch equation by using conformal maps. Indeed, consider an initial doubly-connected domain $D_0 = D_1 \setminus \overline{D_2}$, with D_1 and D_2 are two simply-connected domains close to the discs of radii 1 and b respectively. We introduce for $j \in \{1,2\}$ the conformal mappings $\Phi_j : \mathbb{D}^c \to D_j^c$ taking the form

$$\Phi_1(z) = z + f_1(z) = z + \sum_{n=0}^{\infty}\frac{a_n}{z^n}, \quad \Phi_2(z) = bz + f_2(z) = bz + \sum_{n=0}^{\infty}\frac{b_n}{z^n}.$$

Thus, from the contour dynamics equation, rotating doubly-connected V-states amounts to finding non-trivial zeros of the non-linear functional $G = (G_1, G_2)$, defined for $j \in \{1,2\}$ and $w \in \mathbb{T}$ by

$$G_j(\lambda,b,\Omega,f_1,f_2)(w) := \mathrm{Im}\left\{\Big(\Omega\Phi_j(w) + S(\lambda,\Phi_2,\Phi_j)(w) - S(\lambda,\Phi_1,\Phi_j)(w)\Big)\overline{w}\overline{\Phi_j'(w)}\right\},$$

with

$$\forall w \in \mathbb{T}, \quad S(\lambda,\Phi_i,\Phi_j)(w) := \fint_{\mathbb{T}} \Phi_i'(\tau)K_0\big(\lambda|\Phi_j(w) - \Phi_i(\tau)|\big)d\tau.$$

For this aim, we shall implement Crandall-Rabinowitz's Theorem, starting from the elementary observation that the annulus A_b defined by (1.5) generates a trivial line of solutions for any $\Omega \in \mathbb{R}$, which will play the role of the bifurcation parameter. In the same section together with the Appendix B, we also study the regularity of G and prove that it is of class C^1 with respect to the functional spaces introduced in Section 2.2. Then, in Section 3, we compute the linearized operator at the equilibrium state and prove that it is a Fourier matrix multiplier. More precisely, for

$$\forall w \in \mathbb{T}, \quad h_1(w) = \sum_{n=0}^{\infty} a_n\overline{w}^n \quad \text{and} \quad h_2(w) = \sum_{n=0}^{\infty} b_n\overline{w}^n,$$

we have

$$DG(\lambda,b,\Omega,0,0)[h_1,h_2](w)=\sum_{n=0}^{\infty}(n+1)M_{n+1}(\lambda,b,\Omega)\begin{pmatrix}a_n\\ b_n\end{pmatrix}\mathrm{Im}(w^{n+1}),$$

where

$$M_n(\lambda,b,\Omega):=\begin{pmatrix}\Omega_n(\lambda)-\Omega-b\Lambda_1(\lambda,b) & b\Lambda_n(\lambda,b)\\ -\Lambda_n(\lambda,b) & \Lambda_1(\lambda,b)-b\big[\Omega_n(\lambda b)+\Omega\big]\end{pmatrix}.$$

We refer to Proposition 3.1 for more details and point out that some difficulties appear there when computing some integrals related to Bessel functions. Then, the kernel for the linearized operator $DG(\lambda,b,\Omega,0,0)$ is non trivial for $\Omega=\Omega^{\pm}_{\mathbf{m}}(\lambda,b)$, as defined in Theorem 1.1, with $\mathbf{m}$ large enough. The restriction to higher symmetry $\mathbf{m}\geqslant N(\lambda,b)$ is needed first to ensure the condition

$$\Delta_{\mathbf{m}}(\lambda,b):=\Big(b\big[\Omega_{\mathbf{m}}(\lambda)+\Omega_{\mathbf{m}}(\lambda b)\big]-(1+b^2)\Lambda_1(\lambda,b)\Big)^2-4b^2\Lambda^2_{\mathbf{m}}(\lambda,b)>0,$$

required in the transversality condition of Crandall-Rabinowitz's Theorem and second to get the monotonicity of the sequences $\big(\Omega^{\pm}_n(\lambda,b)\big)_{n\geqslant N(\lambda,b)}$ (to get a one-dimensional kernel), obtained from tricky asymptotic analysis on the modified Bessel functions. For more details, we refer to Proposition 4.1. We point out that the degenerate case corresponding to $\Delta_{\mathbf{m}}(\lambda,b)=0$ where the transversality is no longer true was studied in [26] for Euler equations ($\lambda=0$). It requires to expand the functional at higher order in order to understand the local structure of the bifurcation diagram. In our case, the dependance of $\Delta_{\mathbf{m}}(\lambda,b)$ with respect to the parameter b is more involved and similar approach may be implemented with a high computational cost. The previous bifurcation occurs a priori in $C^{1+\alpha}$ regularity, but using an elliptic regularity argument, we prove in Lemma 4.1 the analyticity of the boundary for these V-states.

2 Functional settings

In this section, we shall reformulate the problem of finding V-states looking at the zeros of a nonlinear functional G. We also introduce the function spaces used in the analysis and study some regularity aspects for the functional G with respect to these functions spaces.

2.1 Boundary equations

In this subsection we shall obtain the system governing the patch motion. The starting point is the vortex patch equation (1.3), which writes using the complex notation

$$\mathrm{Im}\Big\{\big[\partial_t\gamma(t,s)-\mathbf{v}(t,\gamma(t,s))\big]\overline{\partial_s\gamma(t,s)}\Big\}=0,\tag{2.1}$$

where $s\mapsto\gamma(t,s)$ is a parametrization of the boundary of D_t. Assuming that the patch is uniformly rotating with an angular velocity Ω, we can choose a parametrization γ in the form

$$\gamma(t,s)=e^{\mathrm{i}\Omega t}\gamma(0,s).\tag{2.2}$$

One readily has

$$\mathrm{Im}\Big\{\partial_t\gamma(t,s)\overline{\partial_s\gamma(t,s)}\Big\}=\Omega\mathrm{Re}\Big\{\gamma(0,s)\overline{\partial_s\gamma(0,s)}\Big\}.\tag{2.3}$$

Now, to study the second term in the equation (2.1), one needs an explicit formulation of the velocity field $\mathbf{v}$. It has been proved in [9, 30] that the velocity field associated to $(QGSW)_\lambda$ equations writes in the context of vortex patches as an integral on the boundary, namely

$$\mathbf{v}(t,z)=\frac{1}{2\pi}\int_{\partial D_t}K_0(\lambda|z-\xi|)d\xi,\tag{2.4}$$

where the domain D_t is oriented with the convention "matter on the left" due to Stokes' Theorem and where K_0 is the modified Bessel function of second kind. We shall refer to Appendix A for the definitions and properties of modified Bessel functions. By using (2.2), we obtain

$$\begin{aligned}\mathbf{v}(t,\gamma(t,s)) &= \frac{1}{2\pi}\int_{\partial D_t} K_0\left(\lambda|\gamma(t,s)-\xi|\right)d\xi\\ &= \frac{1}{2\pi}\int_0^1 K_0\left(\lambda|e^{\mathrm{i}\Omega t}\gamma(0,s)-e^{\mathrm{i}\Omega t}\gamma(0,s')|\right)\partial_{s'}\gamma(t,s')ds'\\ &= \frac{e^{\mathrm{i}\Omega t}}{2\pi}\int_0^1 K_0\left(\lambda|\gamma(0,s)-\gamma(0,s')|\right)\partial_{s'}\gamma(0,s')ds'\\ &= \frac{e^{\mathrm{i}\Omega t}}{2\pi}\int_{\partial D_0} K_0\left(\lambda|\gamma(0,s)-\xi|\right)d\xi\\ &= e^{\mathrm{i}\Omega t}\mathbf{v}(0,\gamma(0,s)).\end{aligned}$$

Consequently using again (2.2), we get

$$\mathrm{Im}\left\{\mathbf{v}(t,\gamma(t,s))\overline{\partial_s\gamma(t,s)}\right\} = \mathrm{Im}\left\{\mathbf{v}(0,\gamma(0,s))\overline{\partial_s\gamma(0,s)}\right\}. \tag{2.5}$$

Putting together (2.3) and (2.5), the equation (2.1) can be rewritten

$$\Omega\mathrm{Re}\left\{\gamma(0,s)\overline{\partial_s\gamma(0,s)}\right\} = \mathrm{Im}\left\{\mathbf{v}(0,\gamma(0,s))\overline{\partial_s\gamma(0,s)}\right\}. \tag{2.6}$$

Let us assume that our starting domain D_0 is doubly-connected, that is

$$D_0 = D_1\backslash\overline{D_2} \quad \text{with} \quad \overline{D}_2\subset D_1,$$

where D_1 and D_2 are simply-connected bounded open domains of $\mathbb{C}$. Then combining (2.4) and (2.6), one should obtain for all $z\in\partial D_0=\partial D_1\cup\partial D_2$,

$$\begin{aligned}\Omega\mathrm{Re}\left\{z\overline{z'}\right\} &= \mathrm{Im}\left\{\frac{1}{2\pi}\int_{\partial D_0}K_0\left(\lambda|z-\xi|\right)d\xi\overline{z'}\right\}\\ &= \mathrm{Im}\left\{\left(\frac{1}{2\pi}\int_{\partial D_1}K_0\left(\lambda|z-\xi|\right)d\xi-\frac{1}{2\pi}\int_{\partial D_2}K_0\left(\lambda|z-\xi|\right)d\xi\right)\overline{z'}\right\},\end{aligned} \tag{2.7}$$

where z' denotes a tangent vector to the boundary ∂D_0 at the point z. The minus sign in front of the integral on ∂D_2 is here because of the orientation convention for the application of Stokes' Theorem. Following the works initiated by Burbea, see for instance [5, 9, 28, 29], we should give the equation(s) to solve by using conformal mappings. For this purpose, we shall recall Riemann mapping Theorem.

Theorem 2.1 (Riemann Mapping). *Let $\mathbb{D}$ denote the unit open ball and $D_0\subset\mathbb{C}$ be a simply connected bounded domain. Then there exists a unique bi-holomorphic map called also conformal map, $\Phi:\mathbb{C}\backslash\overline{\mathbb{D}}\to\mathbb{C}\backslash\overline{D_0}$ taking the form*

$$\Phi(z) = az+\sum_{n=0}^{\infty}\frac{a_n}{z^n},$$

with $a>0$ and $(a_n)_{n\in\mathbb{N}}\in\mathbb{C}^{\mathbb{N}}$.

Notice that in the previous theorem, the domain is only assumed to be simply-connected and bounded. In particular, the existence of the conformal mapping does not depend on the regularity of the boundary. However, information on the regularity of the conformal mapping implies some regularity of the boundary. This is given by the following result which can be found in [40] or in [37, Thm. 3.6].

Theorem 2.2 (Kellogg-Warschawski). *We keep the notations of Riemann mapping Theorem. If the conformal map $\Phi:\mathbb{C}\backslash\overline{\mathbb{D}}\to\mathbb{C}\backslash\overline{D_0}$ has a continuous extension to $\mathbb{C}\backslash\mathbb{D}$ which is of class $C^{n+1+\beta}$ with $n\in\mathbb{N}$ and $\beta\in(0,1)$, then the boundary $\Phi(\mathbb{T})$ is a Jordan curve of class $C^{n+1+\beta}$.*

Assuming that D_1 and D_2 are respectively small deformations of the discs of radii 1 and b, so that the shape of D_0 is close to the annulus A_b defined in (1.5), we shall consider the parametrizations by the conformal mapping $\Phi_j:\mathbb{C}\backslash\overline{\mathbb{D}}\to\mathbb{C}\backslash\overline{D_j}$ satisfying

$$\Phi_1(z) = z+f_1(z) = z\left(1+\sum_{n=1}^{\infty}\frac{a_n}{z^n}\right)$$

and

$$\Phi_2(z) = bz + f_2(z) = z\left(b + \sum_{n=1}^{\infty}\frac{b_n}{z^n}\right).$$

We shall now rewrite the equations by using the conformal parametrizations Φ_1 and Φ_2. First remark that for $w \in \mathbb{T}$, a tangent vector on the boundary ∂D_j at the point $z = \Phi_j(w)$ is given by

$$\overline{z'} = -\mathrm{i}\overline{w}\overline{\Phi_j'(w)}.$$

Inserting this into (2.7) and using the change of variables $\xi = \Phi_j(\tau)$ gives

$$\forall j \in \{1,2\}, \quad \forall w \in \mathbb{T}, \quad G_j(\lambda, b, \Omega, f_1, f_2)(w) = 0,$$

where

$$G_j(\lambda, b, \Omega, f_1, f_2)(w) := \mathrm{Im}\left\{\Big(\Omega\Phi_j(w) + S(\lambda, \Phi_2, \Phi_j)(w) - S(\lambda, \Phi_1, \Phi_j)(w)\Big)\overline{w}\overline{\Phi_j'(w)}\right\}, \tag{2.8}$$

with

$$\forall (i,j) \in \{1,2\}^2, \quad \forall w \in \mathbb{T}, \quad S(\lambda, \Phi_i, \Phi_j)(w) := \fint_{\mathbb{T}} \Phi_i'(\tau) K_0\big(\lambda|\Phi_j(w) - \Phi_i(\tau)|\big)d\tau. \tag{2.9}$$

Then, finding a non trivial uniformly rotating vortex patch for (1.1) reduces to finding zeros of the non-linear functional

$$G := (G_1, G_2).$$

As stated in the introduction, these non trivial solutions may be obtained by bifurcation techniques from trivial solutions which are annuli. Let us recover with this formalism that indeed the annuli rotate for any angular velocity. This is given by the following result.

Lemma 2.1. *Let $b \in (0,1)$. Then the annulus A_b defined in* (1.5) *is a rotating patch for* (1.1) *for any angular velocity $\Omega \in \mathbb{R}$.*

Proof. Taking $f_1 = f_2 = 0$ by in (2.8), we get

$$G_1(\lambda, b, \Omega, 0, 0)(w) = \mathrm{Im}\left\{b\overline{w}\fint_{\mathbb{T}} K_0\left(\lambda|w - b\tau|\right)d\tau - \overline{w}\fint_{\mathbb{T}} K_0\left(\lambda|w - \tau|\right)d\tau\right\}.$$

Using the changes of variables $\tau \mapsto w\tau$ and the fact that $|w| = 1$, we have

$$G_1(\lambda, b, \Omega, 0, 0)(w) = \mathrm{Im}\left\{b\fint_{\mathbb{T}} K_0\left(\lambda|1 - b\tau|\right)d\tau - \fint_{\mathbb{T}} K_0\left(\lambda|1 - \tau|\right)d\tau\right\} = 0.$$

Indeed for $a \in \{1, b\}$, we have by (A.3) and the change of variables $\theta \mapsto -\theta$

$$\begin{aligned}\overline{\fint_{\mathbb{T}} K_0\left(\lambda|1 - a\tau|\right)d\tau} &= \overline{\frac{1}{2\pi}\int_0^{2\pi} K_0\left(\lambda|1 - ae^{\mathrm{i}\theta}|\right)e^{\mathrm{i}\theta}d\theta} \\ &= \frac{1}{2\pi}\int_0^{2\pi} K_0\left(\lambda|1 - ae^{\mathrm{i}\theta}|\right)e^{-\mathrm{i}\theta}d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} K_0\left(\lambda|1 - ae^{-\mathrm{i}\theta}|\right)e^{\mathrm{i}\theta}d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} K_0\left(\lambda|1 - ae^{\mathrm{i}\theta}|\right)e^{\mathrm{i}\theta}d\theta \\ &= \fint_{\mathbb{T}} K_0\left(\lambda|1 - a\tau|\right)d\tau. \end{aligned} \tag{2.10}$$

Similarly, we find

$$G_2(\lambda, b, \Omega, 0, 0)(w) = 0.$$

This proves Lemma 2.1. □

2.2 Function spaces and regularity of the functional

We introduce here the function spaces used along this work. Throughout the paper it is more convenient to think of 2π-periodic function $g:\mathbb{R}\to\mathbb{C}$ as a function of the complex variable $w=e^{\mathrm{i}\theta}$. To be more precise, let $f:\mathbb{T}\to\mathbb{R}^2$, be a continuous function, then it can be assimilated to a 2π-periodic function $g:\mathbb{R}\to\mathbb{R}^2$ via the relation

$$f(w)=g(\theta),\quad w=e^{\mathrm{i}\theta}.$$

Hence, when f is smooth enough, we get

$$f'(w):=\frac{df}{dw}=-\mathrm{i}e^{-\mathrm{i}\theta}g'(\theta).$$

Since $\frac{d}{dw}$ and $\frac{d}{d\theta}$ differ only by a smooth factor with modulus one, we shall in the sequel work with $\frac{d}{dw}$ instead of $\frac{d}{d\theta}$ which appears more suitable in the computations. In addition, if f is of class C^1 and has real Fourier coefficients, then we can easily check that

$$\left(\overline{f}\right)'(w)=-\frac{\overline{f'(w)}}{w^2}. \tag{2.11}$$

We shall now recall the definition of Hölder spaces on the unit circle.

Definition 2.1. *Let $\alpha\in(0,1)$.*

(i) We denote by $C^\alpha(\mathbb{T})$ the space of continuous functions f such that

$$\|f\|_{C^\alpha(\mathbb{T})}:=\|f\|_{L^\infty(\mathbb{T})}+\sup_{\substack{(\tau,w)\in\mathbb{T}^2\\ \tau\neq w}}\frac{|f(\tau)-f(w)|}{|\tau-w|^\alpha}<\infty.$$

(ii) We denote by $C^{1+\alpha}(\mathbb{T})$ the space of C^1 functions with α-Hölder continuous derivative

$$\|f\|_{C^{1+\alpha}(\mathbb{T})}:=\|f\|_{L^\infty(\mathbb{T})}+\left\|\frac{df}{dw}\right\|_{C^\alpha(\mathbb{T})}<\infty.$$

For $\alpha\in(0,1)$, we set

$$X^{1+\alpha}:=X_1^{1+\alpha}\times X_1^{1+\alpha}\quad\text{with}\quad X_1^{1+\alpha}:=\left\{f\in C^{1+\alpha}(\mathbb{T})\quad\text{s.t.}\quad\forall w\in\mathbb{T},\,f(w)=\sum_{n=0}^{\infty}f_n\overline{w}^n,\,f_n\in\mathbb{R}\right\}$$

and

$$Y^\alpha:=Y_1^\alpha\times Y_1^\alpha\quad\text{with}\quad Y_1^\alpha:=\left\{g\in C^\alpha(\mathbb{T})\quad\text{s.t.}\quad\forall w\in\mathbb{T},\,g(w)=\sum_{n=1}^{\infty}g_ne_n(w),\,g_n\in\mathbb{R}\right\},$$

where

$$e_n(w):=\mathrm{Im}(w^n).$$

We denote

$$B_r^{1+\alpha}:=\left\{f\in X_1^{1+\alpha}\quad\text{s.t.}\quad\|f\|_{C^{1+\alpha}(\mathbb{T})}<r\right\}.$$

We can encode the **m**-fold structure in the functional spaces by setting

$$X_{\mathbf{m}}^{1+\alpha}:=X_{1,\mathbf{m}}^{1+\alpha}\times X_{1,\mathbf{m}}^{1+\alpha}\quad\text{with}\quad X_{1,\mathbf{m}}^{1+\alpha}:=\left\{f\in X_1^{1+\alpha}\quad\text{s.t.}\quad\forall w\in\mathbb{T},\,f(w)=\sum_{n=1}^{\infty}f_{\mathbf{m}n-1}\overline{w}^{\mathbf{m}n-1}\right\}$$

and

$$Y_{\mathbf{m}}^\alpha:=Y_{1,\mathbf{m}}^\alpha\times Y_{1,\mathbf{m}}^\alpha\quad\text{with}\quad Y_{1,\mathbf{m}}^\alpha:=\left\{g\in Y_1^\alpha\quad\text{s.t.}\quad\forall w\in\mathbb{T},\,g(w)=\sum_{n=1}^{\infty}g_{\mathbf{m}n}e_{\mathbf{m}n}(w)\right\}.$$

The spaces $X^{1+\alpha}$ and $X_{\mathbf{m}}^{1+\alpha}$ (resp. Y^α and $Y_{\mathbf{m}}^\alpha$) are equipped with the strong product topology of $C^{1+\alpha}(\mathbb{T})\times C^{1+\alpha}(\mathbb{T})$ (resp. $C^\alpha(\mathbb{T})\times C^\alpha(\mathbb{T})$). We also denote

$$B_{r,\mathbf{m}}^{1+\alpha}:=\left\{f\in X_{1,\mathbf{m}}^{1+\alpha}\quad\text{s.t.}\quad\|f\|_{C^{1+\alpha}(\mathbb{T})}<r\right\}=B_r^{1+\alpha}\cap X_{1,\mathbf{m}}^{1+\alpha}.$$

Remark 2.1. *Observe that in the previous function spaces, we imposed the Fourier coefficients to be real. This corresponds to considering 1-fold domains symmetric with respect to the real axis. Due to the rotation invariance of the problem, we can always assume that the axis of symmetry is indeed the real axis. Since we shall look for* $\mathbf{m}$*-fold solutions, this choice of function spaces is not really restrictive. Nevertheless, a deeper reason is related to the one dimensional kernel condition to be checked in order to apply the Crandall-Rabinowitz Theorem (see Proposition 4.1). If the coefficients were allowed to be complex, this would imply a real dimension of the kernel strictly bigger than 1, which has to be avoided.*

We shall now investigate the regularity of the nonlinear functional G defined by (2.8). Indeed, Crandall-Rabinowitz's Theorem C.1 requires some regularity assumptions to apply and this is what we check here. The ingredients of the proof are classical and they are postponed to the Appendix B.

Proposition 2.1. *Let* $\lambda > 0$, $b \in (0,1)$, $\alpha \in (0,1)$ *and* $\mathbf{m} \in \mathbb{N}^*$. *There exists* $r > 0$ *such that*

(i) $G(\lambda, b, \cdot, \cdot, \cdot) : \mathbb{R} \times B_r^{1+\alpha} \times B_r^{1+\alpha} \to Y^\alpha$ *is well-defined and of class* C^1.

(ii) The restriction $G(\lambda, b, \cdot, \cdot, \cdot) : \mathbb{R} \times B_{r,\mathbf{m}}^{1+\alpha} \times B_{r,\mathbf{m}}^{1+\alpha} \to Y_{\mathbf{m}}^\alpha$ *is well-defined.*

(iii) The partial derivative $\partial_\Omega DG(\lambda, b, \cdot, \cdot, \cdot) : \mathbb{R} \times B_r^{1+\alpha} \times B_r^{1+\alpha} \to \mathcal{L}(X^{1+\alpha}, Y^\alpha)$ *exists and is continuous.*

3 Spectral study

In this section, we study the linearized operator at the equilibrium state and look for the degeneracy conditions for its kernel.

3.1 Linearized operator

In this subsection, we compute the differential $DG(\lambda, b, \Omega, 0, 0)$ and show that it acts as a Fourier multiplier. More precisely, we prove the following proposition.

Proposition 3.1. *Let* $\lambda > 0$, $b \in (0,1)$ *and* $\alpha \in (0,1)$. *Then for all* $\Omega \in \mathbb{R}$ *and for all* $(h_1, h_2) \in X^{1+\alpha}$, *if we write*

$$h_1(w) = \sum_{n=0}^{\infty} a_n \overline{w}^n \quad \text{and} \quad h_2(w) = \sum_{n=0}^{\infty} b_n \overline{w}^n,$$

we have for all $w \in \mathbb{T}$

$$DG(\lambda, b, \Omega, 0, 0)(h_1, h_2)(w) = \sum_{n=0}^{\infty} (n+1) M_{n+1}(\lambda, b, \Omega) \begin{pmatrix} a_n \\ b_n \end{pmatrix} e_{n+1}(w),$$

where for all $n \in \mathbb{N}^*$, *the matrix* $M_n(\lambda, b, \Omega)$ *is defined by*

$$M_n(\lambda, b, \Omega) := \begin{pmatrix} \Omega_n(\lambda) - \Omega - b\Lambda_1(\lambda, b) & b\Lambda_n(\lambda, b) \\ -\Lambda_n(\lambda, b) & \Lambda_1(\lambda, b) - b\big[\Omega_n(\lambda b) + \Omega\big] \end{pmatrix},$$

with

$$\Lambda_n(\lambda, b) := I_n(\lambda b) K_n(\lambda)$$

and

$$\forall x > 0, \quad \Omega_n(x) := I_1(x) K_1(x) - I_n(x) K_n(x).$$

Recall that the modified Bessel functions I_n *and* K_n *are defined in Appendix A.*

Proof. Since $G = (G_1, G_2)$, then for given $(h_1, h_2) \in X^{1+\alpha}$, we have

$$DG(\lambda, b, \Omega, 0, 0)(h_1, h_2) = \begin{pmatrix} D_{f_1} G_1(\lambda, b, \Omega, 0, 0) h_1 + D_{f_2} G_1(\lambda, b, \Omega, 0, 0) h_2 \\ D_{f_1} G_2(\lambda, b, \Omega, 0, 0) h_1 + D_{f_2} G_2(\lambda, b, \Omega, 0, 0) h_2 \end{pmatrix}. \tag{3.1}$$

But, with the notation introduced in Appendix B, we can write

$$\begin{cases} D_{f_1} G_1(\lambda, b, \Omega, 0, 0) h_1 &= D_{f_1} \mathcal{S}_1(\lambda, b, \Omega, 0) h_1 + D_{f_1} \mathcal{I}_1(\lambda, b, 0, 0) h_1 \\ D_{f_2} G_2(\lambda, b, \Omega, 0, 0) h_2 &= D_{f_2} \mathcal{S}_2(\lambda, b, \Omega, 0) h_2 + D_{f_2} \mathcal{I}_2(\lambda, b, 0, 0) h_2 \\ D_{f_2} G_1(\lambda, b, \Omega, 0, 0) h_2 &= D_{f_2} \mathcal{I}_1(\lambda, b, 0, 0) h_2 \\ D_{f_1} G_2(\lambda, b, \Omega, 0, 0) h_1 &= D_{f_1} \mathcal{I}_2(\lambda, b, 0, 0) h_1. \end{cases} \tag{3.2}$$

We write

$$h_1(w) = \sum_{n=0}^{\infty} a_n \overline{w}^n \quad \text{ and } \quad h_2(w) = \sum_{n=0}^{\infty} b_n \overline{w}^n.$$

It has already been proved in [9, Prop. 5.8] that for all $w \in \mathbb{T}$,

$$D_{f_1}\mathcal{S}_1(\lambda, b, \Omega, 0)h_1(w) = \sum_{n=0}^{\infty}(n+1)\left(\Omega_{n+1}(\lambda) - \Omega\right) a_n e_{n+1}(w), \tag{3.3}$$

where

$$\Omega_n(\lambda) := I_1(\lambda)K_1(\lambda) - I_n(\lambda)K_n(\lambda).$$

By a similar calculus, we get

$$D_{f_2}\mathcal{S}_2(\lambda, b, \Omega, 0)h_2(w) = -\sum_{n=0}^{\infty}(n+1)b\left(\Omega_{n+1}(\lambda b) + \Omega\right) b_n e_{n+1}(w). \tag{3.4}$$

In view of (B.8), we can write

$$D_{f_1}\mathcal{I}_1(\lambda, b, 0, 0)h_1(w) = \mathcal{L}_1(h_1)(w) + \mathcal{L}_2(h_1)(w),$$

with

$$\mathcal{L}_1(h_1)(w) := \operatorname{Im}\left\{\overline{w}\overline{h_1'(w)}b \fint_{\mathbb{T}} K_0\left(\lambda|w - b\tau|\right) d\tau\right\},$$

$$\mathcal{L}_2(h_1)(w) := \operatorname{Im}\left\{\frac{\lambda b}{2}\overline{w}\fint_{\mathbb{T}} K_0'\left(\lambda|w - b\tau|\right)\frac{\overline{h_1(w)}(w - b\tau) + h_1(w)(\overline{w} - b\overline{\tau})}{|w - b\tau|} d\tau\right\}.$$

By using the change of variables $\tau \mapsto w\tau$ and the fact that $|w| = 1$, we deduce

$$\overline{w}\fint_{\mathbb{T}} K_0\left(\lambda|w - b\tau|\right) d\tau = \fint_{\mathbb{T}} K_0\left(\lambda|1 - b\tau|\right) d\tau.$$

Moreover, from (2.10), we know that

$$\fint_{\mathbb{T}} K_0\left(\lambda|1 - b\tau|\right) d\tau \in \mathbb{R}.$$

So using that

$$|1 - be^{\mathrm{i}\theta}| = \left(1 - 2b\cos(\theta) + b^2\right)^{\frac{1}{2}} \quad \text{ with } \quad b \in (0, 1), \tag{3.5}$$

we obtain from (A.3),

$$\begin{aligned}\fint_{\mathbb{T}} K_0\left(\lambda|1 - b\tau|\right) d\tau &= \operatorname{Re}\left\{\frac{1}{2\pi}\int_0^{2\pi} K_0\left(\lambda|1 - be^{\mathrm{i}\theta}|\right) e^{\mathrm{i}\theta} d\theta\right\}\\ &= \frac{1}{2\pi}\int_0^{2\pi} K_0\left(\lambda|1 - be^{\mathrm{i}\theta}|\right)\cos(\theta) d\theta.\end{aligned}$$

Now, by (A.6) and (A.3), one obtains for all $n \in \mathbb{N}^*$,

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} K_0\left(\lambda|1 - be^{\mathrm{i}\theta}|\right)\cos(n\theta) d\theta &= \frac{1}{2\pi}\int_0^{2\pi}\sum_{m=-\infty}^{\infty} I_m(\lambda b)K_m(\lambda)\cos(m\theta)\cos(n\theta) d\theta\\ &= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} I_m(\lambda b)K_m(\lambda)\int_0^{2\pi}\cos(m\theta)\cos(n\theta) d\theta\\ &= I_n(\lambda b)K_n(\lambda).\end{aligned} \tag{3.6}$$

Notice that the inversion of symbols of summation and integration is possible due to the geometric decay at infinity given by (A.11). Then, we deduce by (2.11) that

$$\mathcal{L}_1(h_1)(w) = -\sum_{n=0}^{\infty} nbI_1(\lambda b)K_1(\lambda)a_n e_{n+1}(w).$$

By using the change of variables $\tau \mapsto w\tau$ and the fact that $|w| = 1$, we infer

$$\overline{w} \fint_{\mathbb{T}} K_0'(\lambda|w - b\tau|) \frac{\overline{h_1(w)}(w - b\tau) + h_1(w)(\overline{w} - b\overline{\tau})}{|w - b\tau|} d\tau$$
$$= \fint_{\mathbb{T}} K_0'(\lambda|1 - b\tau|) \frac{\overline{h_1(w)}w(1 - b\tau) + h_1(w)\overline{w}(1 - b\overline{\tau})}{|1 - b\tau|} d\tau.$$

But

$$\fint_{\mathbb{T}} K_0'(\lambda|1 - b\tau|) \frac{h_1(w)\overline{w}(1 - b\overline{\tau})}{|1 - b\tau|} d\tau = \sum_{n=0}^{\infty} a_n \left(\fint_{\mathbb{T}} K_0'(\lambda|1 - b\tau|) \frac{(1 - b\overline{\tau})}{|1 - b\tau|} d\tau \right) \overline{w}^{n+1}$$

and

$$\fint_{\mathbb{T}} K_0'(\lambda|1 - b\tau|) \frac{\overline{h_1(w)}w(1 - b\tau)}{|1 - b\tau|} d\tau = \sum_{n=0}^{\infty} a_n \left(\fint_{\mathbb{T}} K_0'(\lambda|1 - b\tau|) \frac{(1 - b\tau)}{|1 - b\tau|} d\tau \right) w^{n+1}.$$

Moreover, by writing the line integral with the parametrization $\tau = e^{i\theta}$ and making the change of variables $\theta \mapsto -\theta$, we get as in (2.10)

$$\fint_{\mathbb{T}} K_0'(\lambda|1 - b\tau|) \frac{(1 - b\tau)}{|1 - b\tau|} d\tau \in \mathbb{R} \quad \text{and} \quad \fint_{\mathbb{T}} K_0'(\lambda|1 - b\tau|) \frac{(1 - b\overline{\tau})}{|1 - b\tau|} d\tau \in \mathbb{R}.$$

Since $\mathrm{Im}\left(\overline{w}^{n+1}\right) = -\mathrm{Im}\left(w^{n+1}\right)$, we obtain

$$\mathcal{L}_2(h_1)(w) = \sum_{n=0}^{\infty} a_n \left(\frac{\lambda b}{2} \fint_{\mathbb{T}} K_0'(\lambda|1 - b\tau|) \frac{b(\overline{\tau} - \tau)}{|1 - b\tau|} d\tau \right) \mathrm{Im}(w^{n+1}).$$

An integration by parts together with (3.5) and (3.6) gives

$$\begin{aligned}
\frac{\lambda b}{2} \fint_{\mathbb{T}} K_0'(\lambda|1 - b\tau|) \frac{b(\overline{\tau} - \tau)}{|1 - b\tau|} d\tau &= \frac{\lambda b}{4\pi} \int_0^{2\pi} K_0'\left(\lambda|1 - be^{\mathrm{i}\theta}|\right) \frac{b(e^{-\mathrm{i}\theta} - e^{\mathrm{i}\theta})e^{\mathrm{i}\theta}}{|1 - be^{\mathrm{i}\theta}|} d\theta \\
&= \frac{-b}{2\pi} \int_0^{2\pi} K_0\left(\lambda|1 - be^{\mathrm{i}\theta}|\right) e^{\mathrm{i}\theta} d\theta \\
&= \frac{-b}{2\pi} \int_0^{2\pi} K_0\left(\lambda|1 - be^{\mathrm{i}\theta}|\right) \cos(\theta) d\theta \\
&= -bI_1(\lambda b)K_1(\lambda).
\end{aligned}$$

Therefore,

$$\mathcal{L}_2(h_1)(w) = -\sum_{n=0}^{\infty} bI_1(\lambda b)K_1(\lambda)a_n e_{n+1}(w).$$

Finally,

$$D_{f_1}\mathcal{I}_1(\lambda, b, 0, 0)h_1(w) = -\sum_{n=0}^{\infty} b(n+1)I_1(\lambda b)K_1(\lambda)a_n e_{n+1}(w). \tag{3.7}$$

Similar computations taking into acount the modification with b, change of signs and the fact that $|b - e^{i\theta}| = |1 - be^{i\theta}|$ yield

$$D_{f_2}\mathcal{I}_2(\lambda, b, 0, 0)(h_2)(w) = \sum_{n=0}^{\infty} (n+1)I_1(\lambda b)K_1(\lambda)b_n e_{n+1}(w). \tag{3.8}$$

According to (B.9), we can write

$$D_{f_2}\mathcal{I}_1(\lambda, b, 0, 0)h_2(w) = \mathcal{L}_3(h_2)(w) + \mathcal{L}_4(h_2)(w),$$

with

$$\mathcal{L}_3(h_2)(w) := \mathrm{Im}\left\{ \overline{w} \fint_{\mathbb{T}} h_2'(\tau) K_0(\lambda|w - b\tau|) d\tau \right\},$$
$$\mathcal{L}_4(h_2)(w) := -\frac{\lambda b}{2} \mathrm{Im}\left\{ \overline{w} \fint_{\mathbb{T}} K_0'(\lambda|w - b\tau|) \frac{\overline{h_2(\tau)}(w - b\tau) + h_2(\tau)(\overline{w} - b\overline{\tau})}{|w - b\tau|} d\tau \right\}.$$

The change of variables $\tau \mapsto w\tau$ implies

$$\begin{aligned}\mathcal{L}_3(h_2)(w) &= \mathrm{Im}\left\{ \fint_{\mathbb{T}} h_2'(w\tau) K_0\left(\lambda|1-b\tau|\right) d\tau \right\} \\ &= -\sum_{n=0}^{\infty} n b_n \left(\fint_{\mathbb{T}} \overline{\tau}^{n+1} K_0\left(\lambda|1-b\tau|\right) d\tau \right) \mathrm{Im}(\overline{w}^{n+1}) \\ &= \sum_{n=0}^{\infty} n b_n \left(\fint_{\mathbb{T}} \overline{\tau}^{n+1} K_0\left(\lambda|1-b\tau|\right) d\tau \right) e_{n+1}(w).\end{aligned}$$

But by symmetry and (3.6)

$$\begin{aligned}\fint_{\mathbb{T}} \overline{\tau}^{n+1} K_0\left(\lambda|1-b\tau|\right) d\tau &= \frac{1}{2\pi}\int_0^{2\pi} e^{-\mathrm{i}(n+1)\theta} K_0\left(\lambda|1-be^{\mathrm{i}\theta}|\right) e^{\mathrm{i}\theta} d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} K_0\left(\lambda|1-be^{\mathrm{i}\theta}|\right) \cos(n\theta) d\theta \\ &= I_n(\lambda b) K_n(\lambda).\end{aligned}$$

Hence,

$$\mathcal{L}_3(h_2)(w) = \sum_{n=0}^{\infty} n I_n(\lambda b) K_n(\lambda) b_n e_{n+1}(w).$$

By using the change of variables $\tau \mapsto w\tau$ and the fact that $|w|=1$, we have

$$\mathcal{L}_4(h_2)(w) = \frac{-\lambda b}{2} \mathrm{Im}\left\{ \fint_{\mathbb{T}} K_0'\left(\lambda|1-b\tau|\right) \frac{\overline{h_2(w\tau)} w(1-b\tau) + h_2(w\tau)\overline{w}(1-b\overline{\tau})}{|1-b\tau|} d\tau \right\},$$

which also writes

$$\mathcal{L}_4(h_2)(w) = \frac{-\lambda b}{2} \sum_{n=0}^{\infty} b_n \left(\fint_{\mathbb{T}} K_0'\left(\lambda|1-b\tau|\right) \frac{(\tau^n - \overline{\tau}^n) - b(\tau^{n+1} - \overline{\tau}^{n+1})}{|1-b\tau|} d\tau \right) \mathrm{Im}(w^n).$$

We denote

$$\mathtt{I} := \frac{-\lambda b}{2} \fint_{\mathbb{T}} K_0'\left(\lambda|1-b\tau|\right) \frac{(\tau^n - \overline{\tau}^n) - b(\tau^{n+1} - \overline{\tau}^{n+1})}{|1-b\tau|} d\tau.$$

Since $\mathtt{I} \in \mathbb{R}$, we have

$$\begin{aligned}\mathtt{I} &= \frac{-\lambda b}{4\pi} \int_0^{2\pi} K_0'\left(\lambda|1-be^{\mathrm{i}\theta}|\right) \frac{(e^{\mathrm{i}n\theta} - e^{-\mathrm{i}n\theta}) - b(e^{\mathrm{i}(n+1)\theta} - e^{-\mathrm{i}(n+1)\theta})}{|1-be^{\mathrm{i}\theta}|} e^{\mathrm{i}\theta} d\theta \\ &= \frac{\lambda b}{2\pi} \int_0^{2\pi} K_0'\left(\lambda|1-be^{\mathrm{i}\theta}|\right) \frac{\sin(\theta)}{|1-be^{\mathrm{i}\theta}|} (\sin(n\theta) - b\sin((n+1)\theta)) d\theta.\end{aligned}$$

Integrating by parts with (3.5) and using (3.6) yield

$$\begin{aligned}\mathtt{I} &= \frac{1}{2\pi} \int_0^{2\pi} K_0(\lambda|1-be^{\mathrm{i}\theta}|) \left(b(n+1)\cos((n+1)\theta) - n\cos(n\theta)\right) \\ &= b(n+1) I_{n+1}(\lambda b) K_{n+1}(\lambda) - n I_n(\lambda b) K_n(\lambda).\end{aligned}$$

Therefore,

$$D_{f_2}\mathcal{I}_1(\lambda, b, 0, 0)(h_2)(w) = \sum_{n=0}^{\infty} b(n+1) I_{n+1}(\lambda b) K_{n+1}(\lambda) b_n e_{n+1}(w). \tag{3.9}$$

Similar computations taking into acount the modification with b, change of signs and the fact that $|b - e^{i\theta}| = |1 - be^{i\theta}|$ imply

$$D_{f_1}\mathcal{I}_2(\lambda, b, 0, 0)(h_1)(w) = -\sum_{n=0}^{\infty} (n+1) I_{n+1}(\lambda b) K_{n+1}(\lambda) a_n e_{n+1}(w). \tag{3.10}$$

Gathering (3.1), (3.2), (3.7), (3.10), (3.3), (3.9), (3.8) and (3.4), we get the desired result. The proof of Proposition 3.1 is now complete. □

3.2 Asymptotic monotonicity of the eigenvalues

This subsection is devoted to the proof of Proposition 3.2 concerning the asymptotic monotonicity of the eigenvalues needed to ensure the one dimensional kernel assumption of Crandall-Rabinowitz's Theorem. But first, we have to prove their existence and this is the purpose of the following lemma.

Lemma 3.1. *Let $\lambda > 0$ and $b \in (0,1)$. There exists $N_0(\lambda,b) \in \mathbb{N}^*$ such that for all integer $n \geqslant N_0(\lambda,b)$, there exist two angular velocities*

$$\begin{aligned}\Omega_n^{\pm}(\lambda,b) &:= \frac{1-b^2}{2b}\Lambda_1(\lambda,b) + \frac{1}{2}\Big(\Omega_n(\lambda) - \Omega_n(\lambda b)\Big)\\ &\pm \frac{1}{2b}\sqrt{\Big(b\big[\Omega_n(\lambda)+\Omega_n(\lambda b)\big] - (1+b^2)\Lambda_1(\lambda,b)\Big)^2 - 4b^2\Lambda_n^2(\lambda,b)}\end{aligned} \tag{3.11}$$

for which the matrix $M_n\left(\lambda, b, \Omega_n^{\pm}(\lambda,b)\right)$ is singular.

Proof. The determinant of $M_n(\lambda,b,\Omega)$ is

$$\begin{aligned}\det\big(M_n(\lambda,b,\Omega)\big) &= \Big(\Omega_n(\lambda) - \Omega - b\Lambda_1(\lambda,b)\Big)\Big(\Lambda_1(\lambda,b) - b\big[\Omega_n(\lambda b)+\Omega\big]\Big) + b\Lambda_n^2(\lambda,b)\\ &= b\Omega^2 - B_n(\lambda,b)\Omega + C_n(\lambda,b),\end{aligned} \tag{3.12}$$

where

$$\begin{aligned}B_n(\lambda,b) &:= (1-b^2)\Lambda_1(\lambda,b) + b\big[\Omega_n(\lambda) - \Omega_n(\lambda b)\big],\\ C_n(\lambda,b) &:= b\left[\Big(\Lambda_1(\lambda,b) - \frac{1}{b}\Omega_n(\lambda)\Big)\Big(b\Omega_n(\lambda b) - \Lambda_1(\lambda,b)\Big) + \Lambda_n^2(\lambda,b)\right].\end{aligned}$$

It is a polynomial of degree two in Ω which has at most two roots. Let us compute its discriminant. After straightforward computations, we find

$$\begin{aligned}\Delta_n(\lambda,b) &:= B_n^2(\lambda,b) - 4bC_n(\lambda,b)\\ &= \Big(b\big[\Omega_n(\lambda)+\Omega_n(\lambda b)\big] - (1+b^2)\Lambda_1(\lambda,b)\Big)^2 - 4b^2\Lambda_n^2(\lambda,b).\end{aligned} \tag{3.13}$$

Using the asymptotic expansion of large order (A.10), we infer

$$\forall \lambda > 0, \quad \forall b \in (0,1], \quad I_n(\lambda b)K_n(\lambda) \underset{n\to\infty}{\longrightarrow} 0. \tag{3.14}$$

As a consequence,

$$\Delta_n(\lambda,b) \underset{n\to\infty}{\longrightarrow} \Delta_\infty(\lambda,b), \tag{3.15}$$

where

$$\Delta_\infty(\lambda,b) = \delta_\infty^2(\lambda,b) \quad \text{with} \quad \delta_\infty(\lambda,b) := b\big[I_1(\lambda)K_1(\lambda) + I_1(\lambda b)K_1(\lambda b)\big] - (1+b^2)I_1(\lambda b)K_1(\lambda). \tag{3.16}$$

We can rewrite $\delta_\infty(\lambda,b)$ as

$$\delta_\infty(\lambda,b) = \big[bI_1(\lambda) - I_1(\lambda b)\big]K_1(\lambda) + bI_1(\lambda b)\big[K_1(\lambda b) - bK_1(\lambda)\big].$$

According to (A.7) and (A.3), we find $K_1' < 0$ on $(0,\infty)$, which implies in turn the strict decay property of K_1 on $(0,\infty)$. Therefore, since $b \in (0,1)$, we get

$$bK_1(\lambda) < K_1(\lambda) < K_1(\lambda b).$$

Now since $b \in (0,1)$, we obtain from (A.2),

$$I_1(\lambda b) = \sum_{m=0}^{\infty} \frac{\left(\frac{\lambda b}{2}\right)^{1+2m}}{m!\Gamma(m+2)} < b\sum_{m=0}^{\infty} \frac{\left(\frac{\lambda}{2}\right)^{1+2m}}{m!\Gamma(m+2)} = bI_1(\lambda).$$

Finally,

$$\Delta_\infty(\lambda,b) > 0.$$

Thus

$$\exists N_0(\lambda,b) \in \mathbb{N}^*, \quad \forall n \in \mathbb{N}^*, \quad n \geqslant N_0(\lambda,b) \Rightarrow \Delta_n(\lambda,b) > 0. \tag{3.17}$$

Therefore, for $n \geqslant N_0(\lambda, b)$ there exist two angular velocities $\Omega_n^-(\lambda, b)$ and $\Omega_n^+(\lambda, b)$ for which the matrix $M_n(\lambda, b, \Omega_n^{\pm}(\lambda, b))$ is singular. These angular velocities are defined by

$$\begin{aligned}\Omega_n^{\pm}(\lambda, b) &:= \frac{B_n(\lambda, b) \pm \sqrt{\Delta_n(\lambda, b)}}{2b} \\ &= \frac{1-b^2}{2b}\Lambda_1(\lambda, b) + \frac{1}{2}\Big(\Omega_n(\lambda) - \Omega_n(\lambda b)\Big) \\ &\quad \pm \frac{1}{2b}\sqrt{\Big(b\big[\Omega_n(\lambda) + \Omega_n(\lambda b)\big] - (1+b^2)\Lambda_1(\lambda, b)\Big)^2 - 4b^2\Lambda_n^2(\lambda, b)}.\end{aligned}$$

This ends the proof of Lemma 3.1. □

We shall now study the monotonicity of the eigenvalues obtained in Lemma 3.1. This is a crucial point to obtain later the one dimensional condition for the kernel of the linearized operator given by Proposition 3.1.

Proposition 3.2. *Let $\lambda > 0$ and $b \in (0,1)$. There exists $N(\lambda, b) \in \mathbb{N}^*$ with $N(\lambda, b) \geqslant N_0(\lambda, b)$ where $N_0(\lambda, b)$ is defined in Lemma 3.1 such that*

(i) The sequence $(\Omega_n^+(\lambda, b))_{n \geqslant N(\lambda, b)}$ is strictly increasing and converges to $\Omega_\infty^+(\lambda, b) = I_1(\lambda)K_1(\lambda) - b\Lambda_1(\lambda, b)$.

(ii) The sequence $(\Omega_n^-(\lambda, b))_{n \geqslant N(\lambda, b)}$ is strictly decreasing and converges to $\Omega_\infty^-(\lambda, b) = \frac{\Lambda_1(\lambda, b)}{b} - I_1(\lambda b)K_1(\lambda b)$.

Then, we have for all $(m, n) \in (\mathbb{N}^*)^2$ with $N(\lambda, b) \leqslant n < m$,

$$\Omega_\infty^-(\lambda, b) < \Omega_m^-(\lambda, b) < \Omega_n^-(\lambda, b) < \Omega_n^+(\lambda, b) < \Omega_m^+(\lambda, b) < \Omega_\infty^+(\lambda, b).$$

Proof. The convergence is an immediate consequence of (3.11), (3.15), (3.16) and (3.14). Then, we turn to the asymptotic monotonicity. For that purpose, we study the sign of the difference

$$\Omega_{n+1}^{\pm}(\lambda, b) - \Omega_n^{\pm}(\lambda, b) = \frac{1}{2}\Big(\big[\Omega_{n+1}(\lambda) - \Omega_n(\lambda)\big] - \big[\Omega_{n+1}(\lambda b) - \Omega_n(\lambda b)\big]\Big) \pm \frac{1}{2b}\left[\sqrt{\Delta_{n+1}(\lambda, b)} - \sqrt{\Delta_n(\lambda, b)}\right]$$

for n large enough.

► We first study the difference term before the square roots. We can write

$$\begin{aligned}&\big[\Omega_{n+1}(\lambda) - \Omega_{n+1}(\lambda b)\big] - \big[\Omega_n(\lambda) - \Omega_n(\lambda b)\big] \\ &= \big[\Omega_{n+1}(\lambda) - \Omega_n(\lambda)\big] - \big[\Omega_{n+1}(\lambda b) - \Omega_n(\lambda b)\big] \\ &= \big[I_n(\lambda)K_n(\lambda) - I_{n+1}(\lambda)K_{n+1}(\lambda)\big] - \big[I_n(\lambda b)K_n(\lambda b) - I_{n+1}(\lambda b)K_{n+1}(\lambda b)\big] \\ &:= \varphi_n(\lambda) - \varphi_n(\lambda b).\end{aligned}$$

By vitue of (A.11), we deduce

$$I_n(\lambda)K_n(\lambda) \underset{n\to\infty}{=} \frac{1}{2n} - \frac{\lambda^2}{4n^3} + o_\lambda\left(\frac{1}{n^4}\right).$$

Therefore,

$$\begin{aligned}\varphi_n(\lambda) - \varphi_n(\lambda b) &\underset{n\to\infty}{=} \lambda^2(b^2-1)\frac{(n+1)^3 - n^3}{4n^3(n+1)^3} + o_{\lambda,b}\left(\frac{1}{n^4}\right) \\ &\underset{n\to\infty}{=} \frac{3\lambda^2(b^2-1)}{4n^4} + o_{\lambda,b}\left(\frac{1}{n^4}\right).\end{aligned}$$

We conclude that

$$\frac{1}{2}\Big(\big[\Omega_{n+1}(\lambda) - \Omega_n(\lambda)\big] - \big[\Omega_{n+1}(\lambda b) - \Omega_n(\lambda b)\big]\Big) \underset{n\to\infty}{=} O_{\lambda,b}\left(\frac{1}{n^4}\right). \tag{3.18}$$

► The next task is to look at the asymptotic sign of the difference $\sqrt{\Delta_{n+1}(\lambda, b)} - \sqrt{\Delta_n(\lambda, b)}$. We can write

$$\sqrt{\Delta_{n+1}(\lambda, b)} - \sqrt{\Delta_n(\lambda, b)} = \frac{\Delta_{n+1}(\lambda, b) - \Delta_n(\lambda, b)}{\sqrt{\Delta_{n+1}(\lambda, b)} + \sqrt{\Delta_n(\lambda, b)}}$$

with

$$\begin{aligned}\Delta_{n+1}(\lambda, b) - \Delta_n(\lambda, b) = {}& b\Big(\Omega_{n+1}(\lambda) - \Omega_n(\lambda) + \Omega_{n+1}(\lambda b) - \Omega_n(\lambda b)\Big) \\ &\times \Big(b\big[\Omega_{n+1}(\lambda) + \Omega_n(\lambda) + \Omega_{n+1}(\lambda b) + \Omega_n(\lambda b)\big] - 2(1+b^2)\Lambda_1(\lambda, b)\Big) \\ &+ 4b^2\Big(\Lambda_n(\lambda, b) - \Lambda_{n+1}(\lambda, b)\Big)\Big(\Lambda_n(\lambda, b) + \Lambda_{n+1}(\lambda, b)\Big).\end{aligned}$$

By using (A.11), we have

$$\Lambda_n(\lambda,b)\underset{n\to\infty}{=}\frac{b^n}{2n}+\frac{\lambda^2b^n(b^2-1)}{2n^2}+o_{\lambda,b}\left(\frac{b^n}{n^2}\right).$$

Hence, the following asymptotic expansion holds

$$\Lambda_n(\lambda,b)\pm\Lambda_{n+1}(\lambda,b)\underset{n\to\infty}{=}o_{\lambda,b}\left(\frac{1}{n^2}\right).$$

As a consequence,

$$4b^2\Big(\Lambda_n(\lambda,b)-\Lambda_{n+1}(\lambda,b)\Big)\Big(\Lambda_n(\lambda,b)+\Lambda_{n+1}(\lambda,b)\Big)\underset{n\to\infty}{=}o_{\lambda,b}\left(\frac{1}{n^2}\right).\tag{3.19}$$

In addition,

$$b\Big(\Omega_{n+1}(\lambda)-\Omega_n(\lambda)+\Omega_{n+1}(\lambda b)-\Omega_n(\lambda b)\Big)=b\Big(\varphi_n(\lambda)+\varphi_n(\lambda b)\Big)\underset{n\to\infty}{\sim}\frac{b}{n^2}\tag{3.20}$$

and

$$\begin{aligned}&b\Big[\Omega_{n+1}(\lambda)+\Omega_n(\lambda)+\Omega_{n+1}(\lambda b)+\Omega_n(\lambda b)\Big]-2(1+b^2)\Lambda_1(\lambda,b)\\&=2b\Big[I_1(\lambda)K_1(\lambda)+I_1(\lambda b)K_1(\lambda b)\Big]-2(1+b^2)I_1(\lambda b)K_1(\lambda)\\&\qquad-b\Big[I_{n+1}(\lambda)K_{n+1}(\lambda)+I_{n+1}(\lambda b)K_{n+1}(\lambda b)+I_n(\lambda)K_n(\lambda)+I_n(\lambda b)K_n(\lambda b)\Big]\\&\underset{n\to\infty}{\longrightarrow}2\delta_\infty(\lambda,b),\end{aligned}\tag{3.21}$$

where $\delta_\infty(\lambda,b)$ is defined in (3.16). From (3.15), (3.16), (3.19), (3.20) and (3.21), we obtain

$$\sqrt{\Delta_{n+1}(\lambda,b)}-\sqrt{\Delta_n(\lambda,b)}\underset{n\to\infty}{\sim}\frac{b}{n^2}.\tag{3.22}$$

▶ Combining (3.18) and (3.22), we get

$$\Omega^{\pm}_{n+1}(\lambda,b)-\Omega^{\pm}_n(\lambda,b)\underset{n\to\infty}{\sim}\pm\frac{1}{2n^2}.$$

We conclude that there exists $N(\lambda,b)\geqslant N_0(\lambda,b)$ such that

$$\forall n\in\mathbb{N}^*,\quad n\geqslant N(\lambda,b)\Rightarrow\begin{cases}\Omega^+_{n+1}(\lambda,b)-\Omega^+_n(\lambda,b)>0\\\Omega^-_{n+1}(\lambda,b)-\Omega^-_n(\lambda,b)<0,\end{cases}$$

i.e. the sequence $(\Omega_n^+(\lambda,b))_{n\geqslant N(\lambda,b)}$ $\Big($resp. $(\Omega_n^-(\lambda,b))_{n\geqslant N(\lambda,b)}\Big)$ is strictly increasing (resp. decreasing). This achieves the proof of Proposition 3.2. □

We shall now study both important asymptotic behaviours

$$\lambda\to0\quad\text{and}\quad b\to0.$$

The first one corresponds to the Euler case and the second one corresponds to the simply-connected case. We remark that we formally recover (at least partially) [24, Thm. B.] and [9, Thm. 5.1.] looking at these limits. More precisely, we have the following result.

Lemma 3.2. *The spectrum is continuous in the following sense.*

(i) Let $b\in(0,1)$. There exists $\widetilde{N}(b)$ such that

$$\forall n\in\mathbb{N}^*,\ n\geqslant\widetilde{N}(b)\ \Rightarrow\ \Omega_n^{\pm}(\lambda,b)\underset{\lambda\to0}{\longrightarrow}\Omega_n^{\pm}(b),$$

where $\Omega_n^{\pm}(b)$ is defined in (1.6).

(ii) Let $\lambda>0$. There exists $\widetilde{N}(\lambda)$ such that

$$\forall n\in\mathbb{N}^*,\ n\geqslant\widetilde{N}(\lambda)\ \Rightarrow\ \Omega_n^{+}(\lambda,b)\underset{b\to0}{\longrightarrow}\Omega_n(\lambda),$$

where $\Omega_n(\lambda)$ is defined in (1.7).

Proof. **(i)** In view of (A.9), we deduce

$$\forall\, n\in\mathbb{N}^*,\quad \forall\, b\in(0,1],\quad I_n(\lambda b)K_n(\lambda)\underset{\lambda\to 0}{\longrightarrow}\frac{b^n}{2n}. \tag{3.23}$$

In what follows, we fix $b\in(0,1)$. By virtue of (3.23), the matrices M_n defined in Proposition 3.1, satisfy the following convergence

$$\forall n\in\mathbb{N}^*,\quad M_n(\lambda,b,\Omega)\underset{\lambda\to 0}{\longrightarrow}M_n(b,\Omega):=\begin{pmatrix}\frac{n-1}{2n}-\frac{b^2}{2}-\Omega & \frac{b^{n+1}}{2n}\\ -\frac{b^n}{2n} & \frac{b}{2}-\frac{b(n-1)}{2n}-b\Omega\end{pmatrix}.$$

After straightforward computations, we find

$$\det\big(M_n(b,\Omega)\big)=b\Omega^2-\frac{b(1-b^2)}{2}\Omega+\frac{b}{4n^2}\left[n(1-b^2)-1+b^{2n}\right].$$

This polynomial of degree two in Ω has the discriminant

$$\Delta_n(b):=\frac{b^2}{n^2}\left[\left(\frac{n(1-b^2)}{2}-1\right)^2-b^{2n}\right].$$

Thus, provided $\Delta_n(b)>0$, i.e. for

$$1+b^n-\frac{n(1-b^2)}{2}<0, \tag{3.24}$$

we have two roots

$$\Omega_n^{\pm}(b):=\frac{1-b^2}{4}\pm\frac{1}{2n}\sqrt{\left(\frac{n(1-b^2)}{2}-1\right)^2-b^{2n}}.$$

Then, we recover the result found in [24, Thm. B.]. Now, observe that the sequence $n\mapsto 1+b^n-\frac{n(1-b^2)}{2}$ is decreasing. Then there exists $\widetilde{N}(b)\in\mathbb{N}^*$ and $c_0>0$ such that

$$\inf_{\substack{n\in\mathbb{N}^*\\ n\geqslant\widetilde{N}(b)}}\Delta_n(b)\geqslant c_0>0.$$

We use the integral representation (A.8), allowing to write

$$\forall n\in\mathbb{N}^*,\quad I_n(\lambda)K_n(\lambda)-\tfrac{1}{2n}=\tfrac12\int_0^\infty\Big[J_0\left(2\lambda\sinh\left(\tfrac{t}{2}\right)\right)-1\Big]e^{-nt}dt.$$

Now using the integral representation (A.1), we find

$$J_0\left(2\lambda\sinh\left(\tfrac{t}{2}\right)\right)-1=\tfrac{1}{\pi}\int_0^\pi\Big[\cos\left(2\lambda\sinh\left(\tfrac{t}{2}\right)\sin(\theta)\right)-1\Big]d\theta.$$

The classical inequalities

$$\forall x\in\mathbb{R},\quad |\cos(x)-1|\leqslant\tfrac{x^2}{2}\quad\text{and}\quad\sinh(x)\leqslant\tfrac{e^x}{2}$$

provide the following estimate for $t\geqslant 0$

$$\Big|J_0\left(2\lambda\sinh\left(\tfrac{t}{2}\right)\right)-1\Big|\leqslant\lambda^2e^t.$$

We conclude that

$$\forall\lambda>0,\quad\sup_{n\in\mathbb{N}\backslash\{0,1\}}\left|I_n(\lambda)K_n(\lambda)-\tfrac{1}{2n}\right|\leqslant\lambda^2. \tag{3.25}$$

On the other hand, we set for $\varepsilon>0$,

$$K_0^\varepsilon(x)=K_0(\varepsilon x)+\log\left(\tfrac{\varepsilon}{2}\right).$$

Remark that (A.5) implies

$$\lim_{\varepsilon\to 0}K_0^\varepsilon(x)=-\log\left(\tfrac{x}{2}\right)-\boldsymbol{\gamma}.$$

By the dominated convergence theorem, one has

$$\forall n\in\mathbb{N}^*,\quad\lim_{\varepsilon\to0}\int_{\mathbb{T}}K_0^\varepsilon(|1-be^{\mathrm{i}\theta}|)\cos(n\theta)d\eta=-\int_{\mathbb{T}}\log(|1-be^{\mathrm{i}\theta}|)\cos(n\theta)d\theta.$$

Now one obtains from (3.6)

$$\forall n\in\mathbb{N}^*,\quad \int_{\mathbb{T}} K_0^\varepsilon(|1-be^{\mathrm{i}\theta}|)\cos(n\theta)d\eta=\int_{\mathbb{T}} K_0(\varepsilon|1-be^{\mathrm{i}\theta}|)\cos(n\theta)d\theta$$
$$=I_n(\varepsilon b)K_n(\varepsilon).$$

Putting together the last two equality with (3.23) yields

$$\forall n\in\mathbb{N}^*,\quad \int_{\mathbb{T}}\log\big(|1-be^{\mathrm{i}\theta}|\big)d\theta=-\tfrac{b^n}{2n}.$$

Added to (3.6), we have

$$\forall\lambda>0,\quad \forall n\in\mathbb{N}^*,\quad I_n(\lambda b)K_n(\lambda)-\tfrac{b^n}{2n}=\int_{\mathbb{T}}\Big[K_0\big(\lambda|1-be^{\mathrm{i}\theta}|\big)+\log\big(|1-be^{\mathrm{i}\theta}|\big)\Big]\cos(n\theta)d\theta.$$

Then, making appeal to the power series decompositions (A.5) and (A.2), we get

$$\forall\lambda>0,\quad \sup_{n\in\mathbb{N}^*}\big|I_n(\lambda b)K_n(\lambda)-\tfrac{b^n}{2n}\big|\lesssim\max(|\log(\lambda)|,1)\lambda^2. \tag{3.26}$$

Combining (3.13), (3.25), (3.26) and (3.23) one obtains

$$\sup_{n\in\mathbb{N}^*}\big|\Delta_n(\lambda,b)-\Delta_n(b)\big|\underset{\lambda\to0}{\longrightarrow}0.$$

Hence, there exists $\lambda_0(b)>0$ such that

$$\inf_{\lambda\in(0,\lambda_0(b)]}\ \inf_{\substack{n\in\mathbb{N}^*\\ n\geqslant\widetilde{N}(b)}}\Delta_n(\lambda,b)\geqslant\tfrac{c_0}{2}>0.$$

Therefore, we deduce from (3.11) and (3.23) that,

$$\forall n\in\mathbb{N}^*,\quad n\geqslant\widetilde{N}(b)\ \Rightarrow\ \Omega_n^\pm(\lambda,b)\underset{\lambda\to0}{\longrightarrow}\Omega_n^\pm(b).$$

(ii) In what follows, we fix $\lambda>0$. By using the asymptotic (A.9), we find

$$\frac{\Lambda_1(\lambda,b)}{b}\underset{b\to0}{\longrightarrow}\frac{\lambda K_1(\lambda)}{2}\quad\text{and}\quad\forall n\in\mathbb{N}^*,\ \Lambda_n(\lambda,b)\underset{b\to0}{\sim}\tfrac{(\lambda b)^n}{2^n n!}K_n(\lambda).$$

Using the power series decomposition (A.2), the decay property of $\lambda\mapsto I_n(\lambda)K_n(\lambda)$ and the asymptotic (3.23), we get

$$\forall n\in\mathbb{N}^*,\quad \big|I_n(\lambda b)K_n(\lambda)-\tfrac{(\lambda b)^n}{2^n n!}K_n(\lambda)\big|\leqslant b^2I_n(\lambda)K_n(\lambda)\leqslant b^2.$$

Thus, we obtain from (3.13), (3.25) and (3.23)

$$\sup_{n\in\mathbb{N}^*}\Big|\Delta_n(\lambda,b)-b^2\Big[\big(\Omega_n(\lambda)+\tfrac{n-1}{2n}-\tfrac{\lambda K_1(\lambda)}{2}\big)^2-\tfrac{(\lambda b)^{2n}}{2^{2n}(n!)^2}K_n^2(\lambda)\Big]\Big|\underset{b\to0}{\longrightarrow}0. \tag{3.27}$$

Notice that

$$\Omega_n(\lambda)+\tfrac{n-1}{2n}-\tfrac{\lambda K_1(\lambda)}{2}\underset{n\to\infty}{\longrightarrow}I_1(\lambda)K_1(\lambda)+\tfrac{1-\lambda K_1(\lambda)}{2}.$$

Consider the function φ defined by $\forall x>0,\varphi(x)=xK_1(x)$. From (A.4), we get

$$\varphi'(x)=K_1(x)+xK_1'(x)=-xK_0(x)<0.$$

Hence φ is strictly decreasing on $(0,\infty)$. Moreover, in view of the asymptotic (A.9), we infer

$$\lim_{x\to0}\varphi(x)=1.$$

Thus, using also (A.3), we obtain

$$\forall x>0,\quad \varphi(x)\in(0,1).$$

Therefore, we deduce that there exists $\widetilde{N}(\lambda)\in\mathbb{N}^*$ such that

$$\forall n\in\mathbb{N}^*,\quad n\geqslant\widetilde{N}(\lambda)\ \Rightarrow\ \Omega_n(\lambda)+\tfrac{n-1}{2n}-\tfrac{\lambda K_1(\lambda)}{2}>0.$$

In addition, using (A.10) and up to increase the value of $\widetilde{N}(\lambda)$ one gets

$$\forall n\in\mathbb{N}^*,\quad n\geqslant \widetilde{N}(\lambda)\ \Rightarrow\ \tfrac{(\lambda b)^{2n}}{2^{2n}(n!)^2}K_n^2(\lambda)\leqslant 1.$$

Coming back to (3.27), we infer the existence of $b_0(\lambda)\in(0,1)$ such that

$$\forall b\in(0,b_0(\lambda)),\quad \forall n\in\mathbb{N}^*,\quad n\geqslant \widetilde{N}(\lambda)\ \Rightarrow\ \Delta_n(\lambda,b)>0.$$

Thus, we get from (3.11)

$$\forall n\in\mathbb{N}^*,\quad n\geqslant \widetilde{N}(\lambda)\ \Rightarrow\ \Omega_n^+(\lambda,b)\underset{b\to 0}{\longrightarrow}\Omega_n(\lambda).$$

Then, we partially recover the result found in [9, Thm. 5.1.]. We also obtain, up to increase the value of $\widetilde{N}(\lambda)$,

$$\forall n\in\mathbb{N}^*,\quad n\geqslant \widetilde{N}(\lambda)\ \Rightarrow\ \Omega_n^-(\lambda,b)\underset{b\to 0}{\longrightarrow}\Omega_n^-(\lambda):=\frac{\lambda n K_1(\lambda)-n+1}{2n}.$$

Unfortunately, we cannot prove bifurcation from these eigenvalues. □

4 Bifurcation from simple eigenvalues

We prove here the following result which implies the main Theorem 1.1 by a direct application of Crandall-Rabinowitz's Theorem C.1.

Proposition 4.1. *Let $\lambda>0$, $b\in(0,1)$, $\alpha\in(0,1)$ and $\mathbf{m}\in\mathbb{N}^*$ such that $\mathbf{m}\geqslant N(\lambda,b)$. Then the following assertions hold true.*

(i) There exists $r>0$ such that $G(\lambda,b,\cdot,\cdot,\cdot):\mathbb{R}\times B_{r,\mathbf{m}}^{1+\alpha}\times B_{r,\mathbf{m}}^{1+\alpha}\to Y_{\mathbf{m}}^{\alpha}$ is well-defined and of class C^1.

(ii) The kernel $\ker\Big(DG\big(\lambda,b,\Omega_{\mathbf{m}}^{\pm}(\lambda,b),0,0\big)\Big)$ is one-dimensional and generated by

$$\begin{array}{rccl} v_{0,\mathbf{m}}: & \mathbb{T} & \to & \mathbb{C}^2\\ & w & \mapsto & \begin{pmatrix} b\big[\Omega_{\mathbf{m}}(\lambda b)+\Omega_{\mathbf{m}}^{\pm}(\lambda,b)\big]-\Lambda_1(\lambda,b)\\ -\Lambda_{\mathbf{m}}(\lambda,b)\end{pmatrix}\overline{w}^{\mathbf{m}-1}.\end{array}$$

(iii) The range $R\Big(DG\big(\lambda,b,\Omega_{\mathbf{m}}^{\pm}(\lambda,b),0,0\big)\Big)$ is closed and of codimension one in $Y_{\mathbf{m}}^{\alpha}$.

(iv) Tranversality condition :

$$\partial_\Omega DG\big(\lambda,b,\Omega_{\mathbf{m}}^{\pm}(\lambda,b),0,0\big)(v_{0,\mathbf{m}})\notin R\Big(DG\big(\lambda,b,\Omega_{\mathbf{m}}^{\pm}(\lambda,b),0,0\big)\Big).$$

Proof. **(i)** Follows from Proposition 2.1.
(ii) Let $(h_1,h_2)\in X_{\mathbf{m}}^{1+\alpha}$. We write

$$h_1(w)=\sum_{n=1}^{\infty}a_n\overline{w}^{n\mathbf{m}-1}\qquad\text{and}\qquad h_2(w)=\sum_{n=1}^{\infty}b_n\overline{w}^{n\mathbf{m}-1}. \tag{4.1}$$

Proposition 3.1 gives

$$\forall w\in\mathbb{T},\quad DG(\lambda,b,\Omega,0,0)(h_1,h_2)(w)=\sum_{n=1}^{\infty}n\mathbf{m}M_{n\mathbf{m}}(\lambda,b,\Omega)\begin{pmatrix}a_n\\ b_n\end{pmatrix}e_{n\mathbf{m}}(w). \tag{4.2}$$

For $\Omega\in\{\Omega_{\mathbf{m}}^-(\lambda,b),\Omega_{\mathbf{m}}^+(\lambda,b)\}$, we have

$$\det\Big(M_{\mathbf{m}}\big(\lambda,b,\Omega_{\mathbf{m}}^{\pm}(\lambda,b)\big)\Big)=0.$$

Thus, the kernel of $DG\big(\lambda,\Omega_{\mathbf{m}}^{\pm}(\lambda,b),0,0\big)$ is non trivial and it is one dimensional if and only if

$$\forall n\in\mathbb{N}^*,\quad n\geqslant 2\Rightarrow \det\Big(M_{n\mathbf{m}}\big(\lambda,b,\Omega_{\mathbf{m}}^{\pm}(\lambda,b)\big)\Big)\neq 0. \tag{4.3}$$

The previous condition is satisfied in view of Proposition 3.2. Hence, we have the equivalence

$$(h_1,h_2)\in\ker\Big(DG\big(\lambda,b,\Omega_{\mathbf{m}}^{\pm}(\lambda,b),0,0\big)\Big)\Leftrightarrow\begin{cases}\forall n\in\mathbb{N}^*,\quad n\geqslant 2\Rightarrow a_n=0=b_n\\ \begin{pmatrix}a_1\\ b_1\end{pmatrix}\in\ker\Big(M_{\mathbf{m}}\big(\lambda,b,\Omega_{\mathbf{m}}^{\pm}(\lambda,b)\big)\Big).\end{cases}$$

Therefore, we can select as generator of $\ker\Big(DG\big(\lambda,b,\Omega_{\mathbf{m}}^{\pm}(\lambda,b),0,0\big)\Big)$ the following pair of functions

$$\begin{array}{rrcl} v_{0,\mathbf{m}}: & \mathbb{T} & \to & \mathbb{C}^2\\ & w & \mapsto & \begin{pmatrix} b\big[\Omega_{\mathbf{m}}(\lambda b)+\Omega_{\mathbf{m}}^{\pm}(\lambda,b)\big]-\Lambda_1(\lambda,b)\\ -\Lambda_{\mathbf{m}}(\lambda,b)\end{pmatrix}\overline{w}^{\mathbf{m}-1}. \end{array}$$

(iii) We consider the set $Z_{\mathbf{m}}$ defined by

$$Z_{\mathbf{m}}:=\Bigg\{g=(g_1,g_2)\in Y_{\mathbf{m}}^{\alpha}\quad\text{s.t.}\quad\forall w\in\mathbb{T},\quad g(w)=\sum_{n=1}^{\infty}\begin{pmatrix}\mathscr{A}_n\\ \mathscr{B}_n\end{pmatrix}e_{n\mathbf{m}}(w),$$
$$\forall n\in\mathbb{N}^*,\quad(\mathscr{A}_n,\mathscr{B}_n)\in\mathbb{R}^2\quad\text{and}\quad\exists(a_1,b_1)\in\mathbb{R}^2,\quad M_{\mathbf{m}}\big(\lambda,b,\Omega_{\mathbf{m}}^{\pm}(\lambda,b)\big)\begin{pmatrix}a_1\\ b_1\end{pmatrix}=\begin{pmatrix}\mathscr{A}_1\\ \mathscr{B}_1\end{pmatrix}\Bigg\}.$$

Clearly, $Z_{\mathbf{m}}$ is a closed sub-vector space of codimension one in $Y_{\mathbf{m}}^{\alpha}$. It remains to prove that it coincides with the range of $DG\big(\lambda,b,\Omega_{\mathbf{m}}^{\pm}(\lambda,b),0,0\big)$. Obviously, we have the inclusion

$$R\Big(DG\big(\lambda,b,\Omega_{\mathbf{m}}^{\pm}(\lambda,b),0,0\big)\Big)\subset Z_{\mathbf{m}}.$$

We are left to prove the converse inclusion. Let $(g_1,g_2)\in Z_{\mathbf{m}}$. We shall prove that the equation

$$DG\big(\lambda,b,\Omega_{\mathbf{m}}^{\pm}(\lambda,b),0,0\big)(h_1,h_2)=(g_1,g_2)$$

admits a solution $(h_1,h_2)\in X_{\mathbf{m}}^{1+\alpha}$ in the form (4.1). According to (4.2), the previous equation is equivalent to the following countable set of equations

$$\forall n\in\mathbb{N}^*,\quad n\mathbf{m}M_{n\mathbf{m}}\big(\lambda,b,\Omega_{\mathbf{m}}^{\pm}(\lambda,b)\big)\begin{pmatrix}a_n\\ b_n\end{pmatrix}=\begin{pmatrix}\mathscr{A}_n\\ \mathscr{B}_n\end{pmatrix}.$$

For $n=1$, the existence follows from the definition of $Z_{\mathbf{m}}$. Thanks to (4.3), the sequences $(a_n)_{n\geqslant2}$ and $(b_n)_{n\geqslant2}$ are uniquely determined by

$$\forall n\in\mathbb{N}^*,\quad n\geqslant2\Rightarrow\begin{pmatrix}a_n\\ b_n\end{pmatrix}=\frac{1}{n\mathbf{m}}M_{n\mathbf{m}}^{-1}\big(\lambda,b,\Omega_{\mathbf{m}}^{\pm}(\lambda,b)\big)\begin{pmatrix}\mathscr{A}_n\\ \mathscr{B}_n\end{pmatrix},$$

or equivalently,

$$\left\{\begin{array}{rcl} a_n & = & \dfrac{\Lambda_1(\lambda,b)-b\big[\Omega_{n\mathbf{m}}(\lambda b)+\Omega_{\mathbf{m}}^{\pm}(\lambda,b)\big]}{n\mathbf{m}\det\Big(M_{n\mathbf{m}}\big(\lambda,b,\Omega_{\mathbf{m}}^{\pm}(\lambda,b)\big)\Big)}\mathscr{A}_n-\dfrac{b\Lambda_{n\mathbf{m}}(\lambda,b)}{n\mathbf{m}\det\Big(M_{n\mathbf{m}}\big(\lambda,b,\Omega_{\mathbf{m}}^{\pm}(\lambda,b)\big)\Big)}\mathscr{B}_n\\[4ex] b_n & = & \dfrac{\Lambda_{n\mathbf{m}}(\lambda,b)}{n\mathbf{m}\det\Big(M_{n\mathbf{m}}\big(\lambda,b,\Omega_{\mathbf{m}}^{\pm}(\lambda,b)\big)\Big)}\mathscr{A}_n+\dfrac{\Omega_{n\mathbf{m}}(\lambda b)+\Omega_{\mathbf{m}}^{\pm}(\lambda,b)-b\Lambda_1(\lambda,b)}{n\mathbf{m}\det\Big(M_{n\mathbf{m}}\big(\lambda,b,\Omega_{\mathbf{m}}^{\pm}(\lambda,b)\big)\Big)}\mathscr{B}_n. \end{array}\right.$$

It remains to prove the regularity, that is $(h_1,h_2)\in X_{\mathbf{m}}^{1+\alpha}$. For that purpose, we show

$$w\mapsto\begin{pmatrix}h_1(w)-a_1\overline{w}^{\mathbf{m}-1}\\ h_2(w)-a_2\overline{w}^{\mathbf{m}-1}\end{pmatrix}\in C^{1+\alpha}(\mathbb{T})\times C^{1+\alpha}(\mathbb{T}).$$

We may focus on the first component, the second one being analogous. We set

$$H_1(\lambda,b,\mathbf{m})(w):=\sum_{n=2}^{\infty}\frac{\mathscr{A}_n}{n\det\Big(M_{n\mathbf{m}}\big(\lambda,b,\Omega_{\mathbf{m}}^{\pm}(\lambda,b)\big)\Big)}w^n,\quad H_2(w):=\sum_{n=2}^{\infty}\frac{\mathscr{B}_n}{n}w^n$$

and

$$\mathscr{G}_1(\lambda,b,\mathbf{m})(w):=\sum_{n=2}^{\infty}I_{n\mathbf{m}}(\lambda b)K_{n\mathbf{m}}(\lambda b)w^n,\quad\mathscr{G}_2(\lambda,b,\mathbf{m})(w):=\sum_{n=2}^{\infty}\frac{\Lambda_{n\mathbf{m}}(\lambda,b)}{\det\Big(M_{n\mathbf{m}}\big(\lambda,b,\Omega_{\mathbf{m}}^{\pm}(\lambda,b)\big)\Big)}w^n.$$

If we denote $\widetilde{h}_1(w):=h_1(w)-a_1\overline{w}^{\mathbf{m}-1}$, then we can write

$$\begin{aligned}\widetilde{h}_1(w)=&\,C_1(\lambda,b,\mathbf{m})wH_1(\lambda,b,\mathbf{m})\left(\overline{w}^{\mathbf{m}}\right)\\ &+C_2(b,\mathbf{m})w(\mathscr{G}_1(\lambda,b,\mathbf{m})*H_1(\lambda,b,\mathbf{m}))\left(\overline{w}^{\mathbf{m}}\right)\\ &+C_2(b,\mathbf{m})w(\mathscr{G}_2(\lambda,b,\mathbf{m})*H_2)\left(\overline{w}^{\mathbf{m}}\right),\end{aligned}\tag{4.4}$$

where

$$C_1(\lambda, b, \mathbf{m}) := \frac{\Lambda_1(\lambda, b) - b\Omega^{\pm}_{\mathbf{m}}(\lambda, b) - bI_1(\lambda b)K_1(\lambda b)}{\mathbf{m}},$$
$$C_2(b, \mathbf{m}) := -\frac{b}{\mathbf{m}}.$$

The convolution must be understood in the usual sense, that is

$$\forall w = e^{\mathrm{i}\theta} \in \mathbb{T}, \quad f * g(w) = \fint_{\mathbb{T}} f(\tau) g(w\overline{\tau}) \frac{d\tau}{\tau} = \frac{1}{2\pi} \int_0^{2\pi} f\left(e^{\mathrm{i}\eta}\right) g\left(e^{\mathrm{i}(\theta-\eta)}\right) d\eta.$$

We shall use the classical convolution law

$$L^1(\mathbb{T}) * C^{1+\alpha}(\mathbb{T}) \hookrightarrow C^{1+\alpha}(\mathbb{T}). \tag{4.5}$$

By using the decay property of the product $I_n K_n$ and the asymptotic (A.9), we have

$$\|\mathscr{G}_1(\lambda, b, \mathbf{m})\|_{L^1(\mathbb{T})} \lesssim \|\mathscr{G}_1(\lambda, b, \mathbf{m})\|_{L^2(\mathbb{T})} = \left(\sum_{n=2}^{\infty} I^2_{n\mathbf{m}}(\lambda b) K^2_{n\mathbf{m}}(\lambda b)\right)^{\frac{1}{2}} \leqslant \frac{1}{2\mathbf{m}} \left(\sum_{n=2}^{\infty} \frac{1}{n^2}\right)^{\frac{1}{2}} < \infty.$$

We also have

$$\|\mathscr{G}_2(\lambda, b, \mathbf{m})\|_{L^1(\mathbb{T})} \leqslant \|\mathscr{G}_2(\lambda, b, \mathbf{m})\|_{L^\infty(\mathbb{T})} \lesssim \sum_{n=2}^{\infty} b^{n\mathbf{m}} < \infty.$$

Hence

$$\Big(\mathscr{G}_1(\lambda, b, \mathbf{m}), \mathscr{G}_2(\lambda, b, \mathbf{m})\Big) \in \left(L^1(\mathbb{T})\right)^2. \tag{4.6}$$

We now prove that H_1 and H_2 are with regularity $C^{1+\alpha}(\mathbb{T})$.
► *Regularity of H_2 :*
First observe that by Cauchy-Schwarz inequality and the embedding $C^\alpha(\mathbb{T})(\hookrightarrow L^\infty(\mathbb{T})) \hookrightarrow L^2(\mathbb{T})$, we have

$$\begin{aligned}\|H_2\|_{L^\infty(\mathbb{T})} &\leqslant \sum_{n=2}^{\infty} \frac{|\mathscr{B}_n|}{n} \\ &\leqslant \left(\sum_{n=2}^{\infty} \frac{1}{n^2}\right)^{\frac{1}{2}} \left(\sum_{n=2}^{\infty} |\mathscr{B}_n|^2\right)^{\frac{1}{2}} \\ &\lesssim \|g_2\|_{L^2(\mathbb{T})} \\ &\lesssim \|g_2\|_{C^\alpha(\mathbb{T})}.\end{aligned} \tag{4.7}$$

We now have to prove that $H_2' \in C^\alpha(\mathbb{T})$. We show that it coincides, up to slight modifications, with g_2 which is of regularity $C^\alpha(\mathbb{T})$. For that purpose, we show that we can differentiate H_2 term by term.
We denote $(S_N)_{N \geqslant 2}$ (resp. $(R_N)_{N \geqslant 2}$) the sequence of the partial sums (resp. the sequence of the remainders) of the series of functions H_2. One has

$$R_N(w) = \sum_{n=N+1}^{\infty} \frac{\mathscr{B}_n}{n} w^n.$$

Using Cauchy-Schwarz inequality, we obtain similarly to (4.7)

$$\|R_N\|_{L^\infty(\mathbb{T})} \leqslant \left(\sum_{n=N+1}^{\infty} \frac{1}{n^2}\right)^{\frac{1}{2}} \|g_2\|_{C^\alpha(\mathbb{T})} \underset{N \to \infty}{\longrightarrow} 0.$$

Hence

$$\|S_N - H_2\|_{L^\infty(\mathbb{T})} \underset{N \to \infty}{\longrightarrow} 0. \tag{4.8}$$

One has

$$S_N'(w) = \overline{w} \sum_{n=2}^{N} \mathscr{B}_n w^n := \overline{w} g_2^N(w).$$

We set

$$g_2^+(w) := \sum_{n=2}^{\infty} \mathscr{B}_n w^n.$$

By continuity of the Szegö projection defined by

$$\Pi : \sum_{n\in\mathbb{Z}} \alpha_n w^n \mapsto \sum_{n\in\mathbb{N}} \alpha_n w^n$$

from $C^{\alpha}(\mathbb{T})$ into itself (see [17] for more details) added to the fact that $g_2 \in C^{\alpha}(\mathbb{T})$, we deduce that $g_2^+ \in C^{\alpha}(\mathbb{T})$. Applying Bernstein Theorem of Fourier series gives that g_2^+ is the uniform limit of its Fourier series, namely

$$\|S_N' - \overline{w}g_2^+\|_{L^\infty(\mathbb{T})} \underset{N\to\infty}{\longrightarrow} 0. \tag{4.9}$$

Gathering (4.8) and (4.9), we conclude that we can differentiate H_2 term by term and get

$$H_2'(\omega) = \overline{w}g_2^+(w).$$

As a consequence,

$$H_2 \in C^{1+\alpha}(\mathbb{T}). \tag{4.10}$$

► *Regularity of* $H_1(\lambda, b, \mathbf{m})$ *:*
By using (3.12) and (A.11), we have the asymptotic expansion

$$\det\left(M_{n\mathbf{m}}(\lambda, b, \Omega_{\mathbf{m}}^{\pm}(\lambda, b))\right) \underset{n\to\infty}{=} d_\infty(\lambda, b, \mathbf{m}) + \frac{\widetilde{d}_\infty(\lambda, b, \mathbf{m})}{n} + O_{\lambda,b,\mathbf{m}}\left(\frac{1}{n^3}\right), \tag{4.11}$$

with, using Proposition 3.2,

$$\begin{aligned} d_\infty(\lambda, b, \mathbf{m}) &:= \left[I_1(\lambda)K_1(\lambda) - \Omega_{\mathbf{m}}^{\pm}(\lambda, b) - b\Lambda_1(\lambda, b)\right]\left[\Lambda_1(\lambda, b) - b\Omega_{\mathbf{m}}^{\pm}(\lambda, b) - bI_1(\lambda b)K_1(\lambda b)\right] \\ &= b\left[\Omega_\infty^{+}(\lambda, b) - \Omega_{\mathbf{m}}^{\pm}(\lambda, b)\right]\left[\Omega_\infty^{-}(\lambda, b) - \Omega_{\mathbf{m}}^{\pm}(\lambda, b)\right] \\ &< 0 \end{aligned}$$

and, using (3.16),

$$\begin{aligned} \widetilde{d}_\infty(\lambda, b, \mathbf{m}) &:= \frac{b}{2\mathbf{m}}\left[I_1(\lambda)K_1(\lambda) - \Omega_{\mathbf{m}}^{\pm}(\lambda, b) - b\Lambda_1(\lambda, b)\right] - \frac{1}{2\mathbf{m}}\left[\Lambda_1(\lambda, b) - b\Omega_{\mathbf{m}}^{\pm}(\lambda, b) - bI_1(\lambda b)K_1(\lambda b)\right] \\ &= \frac{b\left(I_1(\lambda)K_1(\lambda) + I_1(\lambda b)K_1(\lambda b)\right) - (1+b^2)\Lambda_1(\lambda, b)}{2\mathbf{m}} \\ &= \frac{\delta_\infty(\lambda, b)}{2\mathbf{m}}. \end{aligned}$$

We denote

$$r_n(\lambda, b, \mathbf{m}) := \det\Big(M_{n\mathbf{m}}\big(\lambda, b, \Omega_{\mathbf{m}}^{\pm}(\lambda, b)\big)\Big) - d_\infty(\lambda, b, \mathbf{m}) \underset{n\to\infty}{=} \frac{\widetilde{d}_\infty(\lambda, b, \mathbf{m})}{n} + O_{\lambda,b,\mathbf{m}}\left(\frac{1}{n^3}\right). \tag{4.12}$$

We can write

$$\frac{1}{\det\Big(M_{n\mathbf{m}}\big(\lambda, b, \Omega_{\mathbf{m}}^{\pm}(\lambda, b)\big)\Big)} = \frac{r_n^2(\lambda, b, \mathbf{m})}{d_\infty^2(\lambda, b, \mathbf{m})\det\Big(M_{n\mathbf{m}}\big(\lambda, b, \Omega_{\mathbf{m}}^{\pm}(\lambda, b)\big)\Big)} - \frac{r_n(\lambda, b, \mathbf{m})}{d_\infty^2(\lambda, b, \mathbf{m})} + \frac{1}{d_\infty(\lambda, b, \mathbf{m})}.$$

Thus we can write

$$\begin{aligned} H_1(\lambda, b, \mathbf{m})(w) &= \frac{1}{d_\infty^2(\lambda, b, \mathbf{m})}\sum_{n=2}^{\infty}\frac{\mathscr{A}_n r_n^2(\lambda, b, \mathbf{m})}{n\det\Big(M_{n\mathbf{m}}\big(\lambda, b, \Omega_{\mathbf{m}}^{\pm}(\lambda, b)\big)\Big)}w^n - \frac{1}{d_\infty^2(\lambda, b, \mathbf{m})}\sum_{n=2}^{\infty}\frac{\mathscr{A}_n r_n(\lambda, b, \mathbf{m})}{n}w^n \\ &\quad + \frac{1}{d_\infty(\lambda, b, \mathbf{m})}\sum_{n=2}^{\infty}\frac{\mathscr{A}_n}{n}w^n \\ &:= \frac{1}{d_\infty^2(\lambda, b, \mathbf{m})}H_{1,1}(\lambda, b, \mathbf{m})(w) - \frac{1}{d_\infty^2(\lambda, b, \mathbf{m})}H_{1,2}(\lambda, b, \mathbf{m})(w) + \frac{1}{d_\infty(\lambda, b, \mathbf{m})}H_{1,3}(\lambda, b, \mathbf{m})(w). \end{aligned} \tag{4.13}$$

Now since $(\mathscr{A}_n)_{n\in\mathbb{N}^*} \in l^2(\mathbb{N}^*) \subset l^\infty(\mathbb{N}^*)$, we have

$$\left|\frac{\mathscr{A}_n r_n^2(\lambda, b, \mathbf{m})}{n\det\Big(M_{n\mathbf{m}}\big(\lambda, b, \Omega_{\mathbf{m}}^{\pm}(\lambda, b)\big)\Big)}\right| \underset{n\to\infty}{=} O_{\lambda,b,\mathbf{m}}\left(\frac{1}{n^3}\right).$$

By using the link regularity/decay of Fourier coefficients, we deduce that

$$H_{1,1}(\lambda,b,\mathbf{m})\in C^{1+\alpha}(\mathbb{T}). \tag{4.14}$$

Similarly to (4.10), we can obtain

$$H_{1,3}(\lambda,b,\mathbf{m})\in C^{1+\alpha}(\mathbb{T}). \tag{4.15}$$

By the same method, we can also differentiate term by term $H_{1,2}(\lambda,b,\mathbf{m})$ and obtain

$$\forall w\in\mathbb{T},\quad \big(H_{1,2}(\lambda,b,\mathbf{m})\big)'(w)=\overline{w}\sum_{n=2}^{\infty}\mathscr{A}_n r_n(\lambda,b,\mathbf{m})w^n.$$

Notice that from (4.12), we can write

$$\forall w\in\mathbb{T},\quad w\big(H_{1,2}(\lambda,b,\mathbf{m})\big)'(w)=\widetilde{d}_\infty(\lambda,b,\mathbf{m})H_{1,3}(\lambda,b,\mathbf{m})+(\mathscr{C}*g_1^+)(w),$$

where

$$\forall w\in\mathbb{T},\quad g_1^+(w):=\sum_{n=2}^{\infty}\mathscr{A}_n w^n\quad\text{and}\quad \mathscr{C}(w):=\sum_{n=2}^{\infty}\mathscr{C}_n w^n\quad\text{with}\quad \mathscr{C}_n=O_{\lambda,b,\mathbf{m}}\left(\frac{1}{n^3}\right).$$

Using again the continuity of the Szegö projection, we have

$$g_1^+\in C^{1+\alpha}(\mathbb{T})\subset L^\infty(\mathbb{T})\subset L^1(\mathbb{T})\quad\text{and}\quad \mathscr{C}\in C^{1+\alpha}(\mathbb{T}). \tag{4.16}$$

Using (4.15), (4.16) and (4.5), we deduce that

$$\big(H_{1,2}(\lambda,b,\mathbf{m})\big)'\in C^{1+\alpha}(\mathbb{T})\subset C^{\alpha}(\mathbb{T}).$$

Thus

$$H_{1,2}(\lambda,b,\mathbf{m})\in C^{1+\alpha}(\mathbb{T}). \tag{4.17}$$

Gathering (4.14), (4.17) and (4.15), we conclude that

$$H_1(\lambda,b,\mathbf{m})\in C^{1+\alpha}(\mathbb{T}). \tag{4.18}$$

Putting together (4.4), (4.18), (4.10), (4.6) and (4.5), we finally conclude

$$\widetilde{h}_1\in C^{1+\alpha}(\mathbb{T}).$$

(iv) $\Omega_{\mathbf{m}}^{\pm}(\lambda,b)$ is a simple eigenvalue since $\Delta_{\mathbf{m}}(\lambda,b)>0$. From (B.1) and (B.2), we deduce

$$\begin{cases}\partial_\Omega DG_1\big(\lambda,b,\Omega_{\mathbf{m}}^{\pm}(\lambda,b),0,0\big)(h_1,h_2)(w)=\mathrm{Im}\left\{\overline{h_1'(w)}+\overline{w}h_1(w)\right\}=-\displaystyle\sum_{n=0}^{\infty}n\mathbf{m}a_n e_{n\mathbf{m}}(w)\\ \partial_\Omega DG_2\big(\lambda,b,\Omega_{\mathbf{m}}^{\pm}(\lambda,b),0,0\big)(h_1,h_2)(w)=b\mathrm{Im}\left\{\overline{h_2'(w)}+\overline{w}h_2(w)\right\}=-\displaystyle\sum_{n=0}^{\infty}bn\mathbf{m}b_n e_{n\mathbf{m}}(w).\end{cases}$$

Thus,

$$\partial_\Omega DG\big(\lambda,b,\Omega_{\mathbf{m}}^{\pm}(\lambda,b),0,0\big)(v_{0,\mathbf{m}})(w)=\mathbf{m}\begin{pmatrix}\Lambda_1(\lambda,b)-b\big[\Omega_{\mathbf{m}}(\lambda b)+\Omega_{\mathbf{m}}^{\pm}(\lambda,b)\big]\\ b\Lambda_{\mathbf{m}}(\lambda,b)\end{pmatrix}e_{\mathbf{m}}(w).$$

Notice that the previous expression belongs to the range of $DG\big(\lambda,b,\Omega_{\mathbf{m}}^{\pm}(\lambda,b),0,0\big)$ if and only if the vector

$$\begin{pmatrix}\Lambda_1(\lambda,b)-b\big[\Omega_{\mathbf{m}}(\lambda b)+\Omega_{\mathbf{m}}^{\pm}(\lambda,b)\big]\\ b\Lambda_{\mathbf{m}}(\lambda,b)\end{pmatrix}$$

is a scalar multiple of one column of the matrix $M_{\mathbf{m}}\big(\lambda,b,\Omega_{\mathbf{m}}^{\pm}(\lambda,b)\big)$. This occurs if and only if

$$\Big(\Lambda_1(\lambda,b)-b\big[\Omega_{\mathbf{m}}(\lambda b)+\Omega_{\mathbf{m}}^{\pm}(\lambda,b)\big]\Big)^2-b^2\Lambda_{\mathbf{m}}^2(\lambda,b)=0. \tag{4.19}$$

Putting (4.19) together with $\det\Big(M_{\mathbf{m}}\big(\lambda,b,\Omega_{\mathbf{m}}^{\pm}(\lambda,b)\big)\Big)=0$ implies

$$\Big(\Lambda_1(\lambda,b)-b\big[\Omega_{\mathbf{m}}(\lambda b)+\Omega_{\mathbf{m}}^{\pm}(\lambda,b)\big]\Big)\Big((1-b^2)\Lambda_1(\lambda,b)+b\big[\Omega_{\mathbf{m}}(\lambda)-\Omega_{\mathbf{m}}(\lambda b)\big]-2b\Omega_{\mathbf{m}}^{\pm}(\lambda,b)\Big)=0.$$

Now remark that the above equation is equivalent to

$$\Lambda_1(\lambda,b)-b\big[\Omega_{\mathbf{m}}(\lambda b)+\Omega_{\mathbf{m}}^{\pm}(\lambda,b)\big]=0\quad\text{or}\quad\Omega_{\mathbf{m}}^{\pm}(\lambda,b)=\frac{1}{2b}\Big((1-b^2)\Lambda_1(\lambda,b)+b\big[\Omega_{\mathbf{m}}(\lambda)-\Omega_{\mathbf{m}}(\lambda b)\big]\Big).$$

Since $b\neq 0$ and $\Lambda_{\mathbf{m}}(\lambda,b)\neq 0$, then in view of (4.19), the first equation can't be solved. Then, necessary, the second equation must be satisfied. But we notice that it corresponds to a multiple eigenvalue ($\Delta_{\mathbf{m}}(\lambda,b)=0$), which is excluded here. Therefore, we conclude that

$$\partial_\Omega DG\big(\lambda,b,\Omega_{\mathbf{m}}^{\pm}(\lambda,b),0,0\big)(v_{0,\mathbf{m}})\notin R\Big(DG\big(\lambda,b,\Omega_{\mathbf{m}}^{\pm}(\lambda,b),0,0\big)\Big).$$

This ends the proof of Proposition 4.1. □

The previous proposition allows to construct, for any fixed $\lambda>0$, $b\in(0,1)$, $\alpha\in(0,1)$ and $\mathbf{m}\geqslant N(\lambda,b)$ two branches of $\mathbf{m}$-fold doubly-connected V-states with regularity $C^{1+\alpha}$ bifurcating from the annulus A_b at the angular velocities $\Omega_{\mathbf{m}}^{\pm}(\lambda,b)$ for the $(QGSW)_\lambda$ equations. Actually, we have the following better result for the regularity of the boundary.

Lemma 4.1. *Let $\lambda>0$, $b\in(0,1)$ and $\mathbf{m}\geqslant N(\lambda,b)$. Consider a $\mathbf{m}$-fold doubly-connected V-state close to A_b for $(QGSW)_\lambda$ equations, rotating with an angular velocity Ω and associated with an initial domain $D_0=D_1\setminus\overline{D_2}$, where D_1 and D_2 are simply-connected domains satisfying $\overline{D_2}\subset D_1$ and parametrized by the following conformal mappings*

$$\Phi_1(w)=w+f_1(w),\qquad\Phi_2(w)=bw+f_2(w),\qquad f_1,f_2\in B_{r,\mathbf{m}}^{1+\alpha}.$$

If $r>0$ is small enough, then the boundaries ∂D_1 and ∂D_2 are analytic.

Proof. The proof is done in the spirit of [21, Sec. 5.4] by applying [33, Thm. 3.1']. We highlight that the positive number r quantifies the smallness of f_1 and f_2 in the $C^{1+\alpha}$ topology. We mention that (2.6) can also be written as follows

$$\tfrac{\Omega}{2}\partial_s\big|\gamma(0,s)\big|^2=\partial_s\Big(\mathbf{\Psi}\big(0,\gamma(0,s)\big)\Big),\tag{4.20}$$

where $\mathbf{\Psi}$ is the velocity potential given by

$$\mathbf{v}(t,z)=\nabla^\perp\mathbf{\Psi}(t,z)=2\mathrm{i}\partial_{\overline{z}}\mathbf{\Psi}(t,z),\qquad(\Delta-\lambda^2)\mathbf{\Psi}(t,z)=\mathbf{1}_{D_t}(z).\tag{4.21}$$

Therefore, integrating the relation (4.20), there exists for each $j\in\{1,2\}$ a constant $c_j\in\mathbb{R}$ such that

$$\forall z\in\partial D_j,\quad u_j(z):=\mathbf{\Psi}(0,z)-\tfrac{\Omega}{2}|z|^2-c_j=0.$$

Fix $j\in\{1,2\}$. By compactness of ∂D_j, there exist $M\in\mathbb{N}^*$, $(x_{k,j})_{1\leqslant k\leqslant M}\in(\partial D_j)^M$ and $\varepsilon>0$ (small) such that we can write

$$\partial D_j\subset\bigcup_{k=1}^{M}B(x_{k,j},\varepsilon),\qquad\text{with}\qquad B(x_{k,j},\varepsilon)\cap\partial D_{3-j}=\varnothing.$$

Fix $k\in[\![1,M]\!]$ and denote

$$\Gamma_{k,j}:=B(x_{k,j},\varepsilon)\cap\partial D_j,\qquad\mathcal{O}_{k,j}^{-}:=B(x_{k,j},\varepsilon)\cap D_0,\qquad\mathcal{O}_{k,j}^{+}:=B(x_{k,j},\varepsilon)\cap\big(\mathbb{R}^2\setminus D_0\big).$$

Solving the Helmoltz problem (4.21) as in [30], the stream function writes

$$\mathbf{\Psi}(0,z)=-\frac{1}{2\pi}\int_{D_0}K_0(\lambda|z-\xi|)dA(\xi),$$

where dA denotes the planar Lebesgue measure. From (A.5)-(A.2), we can write

$$\begin{aligned}\mathbf{\Psi}(0,z)&=\frac{1}{2\pi}\int_{D_0}\log(|z-\xi|)dA(\xi)+\int_{D_0}\mathtt{F}(|z-\xi|)dA(\xi)\\&:=\mathbf{\Psi}_1(z)+\mathbf{\Psi}_2(z).\end{aligned}$$

where $\mathtt{F},\mathtt{F}'$ are bounded at 0 and $\mathtt{F}''$ is integrable at the origin. Notice that $\mathbf{\Psi}_1$ corresponds to the classical Euler velocity potential. Since D_0 is of regularity $C^{1+\alpha}$ then one can classically prove that

$$\mathbf{\Psi}_1\in C^{1+\alpha}\big(\mathbb{R}^2,\mathbb{R}\big)\cap C^{2+\alpha}\big(\overline{D_0},\mathbb{R}\big)\cap C^{2+\alpha}\big(\mathbb{R}^2\setminus D_0,\mathbb{R}\big).$$

For instance, the $C^{1+\alpha}$ regularity is obtained by using [14, Exercise 4.8 (a)]. As for the $C^{2+\alpha}$ regularity, one may use in particular the "Main Lemma" in [36] applied to the Calderón-Zygmund type operator $\mathbf{1}_{D_0}\mapsto\nabla\nabla^\perp\mathbf{\Psi}_1$. The term $\mathbf{\Psi}_2$ being less singular, we get

$$\mathbf{\Psi}(0,\cdot)\in C^{1+\alpha}\big(\mathbb{R}^2,\mathbb{R}\big)\cap C^{2+\alpha}\big(\overline{D_0},\mathbb{R}\big)\cap C^{2+\alpha}\big(\mathbb{R}^2\setminus D_0,\mathbb{R}\big)$$

and then

$$u_j \in C^1\big(B(x_{k,j},\varepsilon)\,,\,\mathbb{R}\big) \cap C^2\big(\mathcal{O}^-_{k,j} \cup \Gamma_{k,j}\,,\,\mathbb{R}\big) \cap C^2\big(\mathcal{O}^+_{k,j} \cup \Gamma_{k,j}\,,\,\mathbb{R}\big).$$

One can easily find from (4.21) that

$$\begin{aligned}
&\forall z \in \mathcal{O}^+_{k,j}, \quad 0 = \mathscr{F}_j(z, u_j, Du_j, D^2u_j) := (\Delta - \lambda^2)u_j(z) - \tfrac{\lambda^2}{2}\Omega|z|^2 - \lambda^2 c_j + 2\Omega,\\
&\forall z \in \mathcal{O}^-_{k,j}, \quad 0 = \mathscr{G}_j(z, u_j, Du_j, D^2u_j) := (\Delta - \lambda^2)u_j(z) - \tfrac{\lambda^2}{2}\Omega|z|^2 - \lambda^2 c_j + 2\Omega - 1.
\end{aligned}$$

Observe that the functions $\mathscr{F}_j$ and $\mathscr{G}_j$ are analytic. Thus it remains to prove that

$$\forall z \in \partial D_j, \quad \nabla u_j(z) \cdot \mathbf{n}_j(z) \neq 0, \tag{4.22}$$

where $\mathbf{n}_j$ is a normal unitary vector to ∂D_j. We can write

$$\begin{aligned}
\nabla u_j(z) \cdot \mathbf{n}_j(z) &= \nabla \mathbf{\Psi}(0,z) \cdot \mathbf{n}_j(z) - \Omega z \cdot \mathbf{n}_j(z)\\
&= \nabla^\perp \mathbf{\Psi}(0,z) \cdot \mathbf{i}\mathbf{n}_j(z) - \Omega z \cdot \mathbf{n}_j(z)\\
&= \mathbf{v}(0,z) \cdot \mathbf{i}\mathbf{n}_j(z) - \Omega z \cdot \mathbf{n}_j(z).
\end{aligned} \tag{4.23}$$

The normal unitary vector can be expressed as follows in terms of the conformal mapping

$$\mathbf{n}_j(z) = w\frac{\Phi_j'(w)}{|\Phi_j'(w)|} \qquad \text{if} \qquad z = \Phi_j(w), \qquad w \in \mathbb{T}.$$

On one hand, denoting $b_1 := 1$ and $b_2 := b$, we have for $z = \Phi_j(w) \in \partial D_j$,

$$\begin{aligned}
z \cdot \mathbf{n}_j(z) &= \mathrm{Re}\left\{\Phi_j(w)\overline{w}\frac{\overline{\Phi_j'(w)}}{|\Phi_j'(w)|}\right\}\\
&= b_j + \mathrm{Re}\left\{f_j(w)\overline{w}\frac{\overline{\Phi_j'(w)}}{|\Phi_j'(w)|} + b_j\left(\frac{b_j + \overline{f_j'(w)}}{|b_j + f_j'(w)|} - 1\right)\right\}\\
&= b_j + O(r).
\end{aligned} \tag{4.24}$$

On the other hand,

$$\begin{aligned}
\mathbf{v}(0,z) \cdot \mathbf{i}\mathbf{n}_j(z) &= \mathrm{Re}\left\{\overline{w}\frac{\overline{\Phi_j'(w)}}{|\Phi_j'(w)|}\left(\fint_{\mathbb{T}} \Phi_1'(\tau)K_0\big(\lambda|\Phi_j(w) - \Phi_1(\tau)|\big)d\tau - \fint_{\mathbb{T}} \Phi_2'(\tau)K_0\big(\lambda|\Phi_j(w) - \Phi_2(\tau)|\big)d\tau\right)\right\}\\
&= \mathrm{Re}\left\{\overline{w}\left(\fint_{\mathbb{T}} K_0\big(\lambda|b_jw - \tau|\big)d\tau - b\fint_{\mathbb{T}} K_0\big(\lambda|b_jw - b\tau|\big)d\tau\right)\right\} + \mathscr{J}_1 + \mathscr{J}_2 + \mathscr{J}_3,
\end{aligned} \tag{4.25}$$

where

$$\begin{aligned}
\mathscr{J}_1 &:= \mathrm{Re}\left\{\overline{w}\left(\fint_{\mathbb{T}} K_0\big(\lambda|b_jw - \tau|\big)d\tau - \fint_{\mathbb{T}} K_0\big(\lambda|\Phi_j(w) - \Phi_1(\tau)|\big)d\tau\right)\right\}\\
&\quad - \mathrm{Re}\left\{\overline{w}b\left(\fint_{\mathbb{T}} K_0\big(\lambda|b_jw - b\tau|\big)d\tau - \fint_{\mathbb{T}} K_0\big(\lambda|\Phi_j(w) - \Phi_2(\tau)|\big)d\tau\right)\right\},\\
\mathscr{J}_2 &:= \mathrm{Re}\left\{\overline{w}\left(\frac{b_j + \overline{f_j'(w)}}{|b_j + f_j'(w)|} - 1\right)\left(\fint_{\mathbb{T}} K_0\big(\lambda|\Phi_j(w) - \Phi_1(\tau)|\big)d\tau - b\fint_{\mathbb{T}} K_0\big(\lambda|\Phi_j(w) - \Phi_2(\tau)|\big)d\tau\right)\right\},\\
\mathscr{J}_3 &:= \mathrm{Re}\left\{\overline{w}\frac{\overline{\Phi_j'(w)}}{|\Phi_j'(w)|}\left(\fint_{\mathbb{T}} f_1'(\tau)K_0\big(\lambda|\Phi_j(w) - \Phi_1(\tau)|\big)d\tau - \fint_{\mathbb{T}} f_2'(\tau)K_0\big(\lambda|\Phi_j(w) - \Phi_2(\tau)|\big)d\tau\right)\right\}.
\end{aligned}$$

We shall now prove that the terms $\mathscr{J}_1$, $\mathscr{J}_2$ and $\mathscr{J}_3$ are small. Let us start with $\mathscr{J}_3$. Recalling the notation (B.3), one has

$$|\mathscr{J}_3| \leqslant \|\mathcal{T}_{1j}f_1'\|_{L^\infty(\mathbb{T})} + \|\mathcal{T}_{2j}f_2'\|_{L^\infty(\mathbb{T})}. \tag{4.26}$$

From (B.6), we get

$$\forall (i,j) \in \{1,2\}^2, \qquad i \neq j \quad \Rightarrow \quad \|\mathcal{T}_{ij}f_i'\|_{L^\infty(\mathbb{T})} \lesssim \|f_i'\|_{L^\infty(\mathbb{T})} \lesssim \|f_i\|_{C^{1+\alpha}(\mathbb{T})} \lesssim r. \tag{4.27}$$

Now fix $i \in \{1,2\}$ and denote

$$\mathcal{K}_i(w,\tau) := K_0(\lambda|\Phi_i(w) - \Phi_i(\tau)|).$$

We mention that the triangle inequality and the mean value theorem imply that Φ_i is bi-Lipschitz, namely

$$(1-r)|w-\tau| \leqslant |\Phi_i(w)-\Phi_i(\tau)| \leqslant (1+r)|w-\tau|. \tag{4.28}$$

Recall that K_0 behaves like a logarithm at 0 and using (A.5) we can write

$$K_0'(z) = -\frac{1}{z} + \mathtt{G}(z), \qquad \mathtt{G} \text{ bounded at } 0. \tag{4.29}$$

Therefore, for any $\delta \in (0,1)$, we have

$$|\mathcal{K}_i(w,\tau)| \lesssim \frac{1}{|w-\tau|^\delta} \quad \text{and} \quad |\partial_w \mathcal{K}_i(w,\tau)| \lesssim \frac{1}{|w-\tau|^{1+\delta}}.$$

Thus, applying [17, Lem. 1], we infer

$$\|\mathcal{T}_{ii} f_i'\|_{L^\infty(\mathbb{T})} \lesssim \|f_i'\|_{L^\infty(\mathbb{T})} \lesssim r. \tag{4.30}$$

Putting together (4.26), (4.27) and (4.30), we deduce

$$|\mathscr{J}_3| \lesssim r. \tag{4.31}$$

From the previous computations, one also obtains

$$|\mathscr{J}_2| \lesssim r. \tag{4.32}$$

As for $\mathscr{J}_1$, we may use Taylor formula to write

$$\begin{aligned} &K_0\big(\lambda|\Phi_j(w)-\Phi_i(\tau)|\big) - K_0\big(\lambda|b_j w - b_i\tau|\big) \\ &\quad = \lambda\big(|\Phi_j(w)-\Phi_i(\tau)| - |b_j w - b_i \tau|\big)\int_0^1 K_0'\Big(\lambda|b_j w - b_i\tau| + \lambda t\big(|\Phi_j(w)-\Phi_i(\tau)| - |b_j w - b_i\tau|\big)\Big)dt. \end{aligned}$$

The triangular inequality and the mean value theorem imply

$$\Big||\Phi_j(w)-\Phi_i(\tau)| - |b_j w - b_i\tau|\Big| \leqslant |f_j(w)-f_i(\tau)| \leqslant \begin{cases} 2r & \text{if } i \neq j, \\ r|w-\tau| & \text{if } i = j. \end{cases}$$

Hence using (4.29), (4.28), (B.4) and (B.5), we deduce

$$|\mathscr{J}_1| \lesssim r. \tag{4.33}$$

Moreover, according to the computations carried out in Proposition 3.1 (see also [30, Lem. 3.2]), we have

$$\overline{w}\fint_{\mathbb{T}} K_0(\lambda|z-\tau|)d\tau = I_1(\lambda)K_1(\lambda), \qquad \overline{w}\fint_{\mathbb{T}} K_0(\lambda|w-b\tau|)d\tau = \overline{w}\fint_{\mathbb{T}} K_0(\lambda|bw-\tau|)d\tau = \Lambda_1(\lambda,b). \tag{4.34}$$

Therefore, in view of (4.25), (4.31), (4.32), (4.33), (4.34) and Proposition 3.2, we infer

$$\begin{aligned} \forall z \in \partial D_1, \quad \mathbf{v}(0,z)\cdot \mathbf{in}_1(z) &= I_1(\lambda)K_1(\lambda) - b\Lambda_1(\lambda,b) + O(r) \\ &= \Omega_\infty^+(\lambda,b) + O(r) \end{aligned} \tag{4.35}$$

and

$$\begin{aligned} \forall z \in \partial D_2, \quad \mathbf{v}(0,z)\cdot \mathbf{in}_2(z) &= \Lambda_1(\lambda,b) - bI_1(\lambda b)K_1(\lambda b) + O(r) \\ &= b\Omega_\infty^-(\lambda,b) + O(r). \end{aligned} \tag{4.36}$$

Combining (4.23), (4.24), (4.35) and (4.36), we deduce by triangular inequality

$$\forall z \in \partial D_1, \quad |\nabla u_1(z)\cdot n_1(z)| \geqslant |\Omega_\infty^+(\lambda,b) - \Omega| - Cr$$

and

$$\forall z \in \partial D_2, \quad |\nabla u_2(z)\cdot n_2(z)| \geqslant b|\Omega_\infty^-(\lambda,b) - \Omega| - Cr.$$

The Crandall-Rabinowitz Theorem implies that Ω is close to $\Omega_{\mathbf{m}}^\pm(\lambda,b)$. Hence, according to Proposition 3.2, we can say

$$\Omega_\infty^\pm(\lambda,b) - \Omega \neq 0.$$

Thus, up to take r sufficiently small, we get (4.22). Consequently, $\Gamma_{k,j}$ is analytic from which we deduce by reconstruction that ∂D_j is also analytic. $\square$

A Formulae on modified Bessel functions

We shall collect some useful information on modified Bessel functions. For more details we refer to [1, 41]. We define first the Bessel functions of order $\nu \in \mathbb{C}$ by

$$J_\nu(z) = \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{z}{2}\right)^{\nu+2m}}{m!\Gamma(\nu+m+1)}, \quad |\arg(z)| < \pi.$$

Notice that when $\nu \in \mathbb{N}$ we have the following integral representation, see [35, p. 115].

$$J_\nu(x) = \frac{1}{\pi}\int_0^\pi \cos\big(x\sin\theta - \nu\theta\big)d\theta. \tag{A.1}$$

We define the Bessel functions of imaginary argument by

$$I_\nu(z) = \sum_{m=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{\nu+2m}}{m!\Gamma(\nu+m+1)}, \quad |\arg(z)| < \pi \tag{A.2}$$

and

$$K_\nu(z) = \frac{\pi}{2}\frac{I_{-\nu}(z) - I_\nu(z)}{\sin(\nu\pi)}, \quad \nu \in \mathbb{C}\backslash\mathbb{Z}, \quad |\arg(z)| < \pi.$$

For $n \in \mathbb{Z}$, we define $K_n(z) = \lim_{\nu\to n} K_\nu(z)$. We give now useful properties of modified Bessel functions.

Symmetry and positivity properties (see [1, p. 375]) **:**

$$\forall n \in \mathbb{N}, \quad \forall\lambda > 0, \quad I_{-n}(\lambda) = I_n(\lambda) > 0 \quad \text{and} \quad K_{-n}(\lambda) = K_n(\lambda) > 0. \tag{A.3}$$

Derivatives (see [1, p. 376]) **:**

If we set $\mathcal{Z}_\nu(z) = I_\nu(z)$ or $e^{i\nu\pi}K_\nu(z)$, then for all $\nu \in \mathbb{R}$, we have

$$\mathcal{Z}_\nu'(z) = \mathcal{Z}_{\nu-1}(z) - \frac{\nu}{z}\mathcal{Z}_\nu(z) = \mathcal{Z}_{\nu+1}(z) + \frac{\nu}{z}\mathcal{Z}_\nu(z). \tag{A.4}$$

Power series extension for K_n (see [1, p. 375]) **:**

$$\begin{aligned} K_n(z) =& \frac{1}{2}\left(\frac{z}{2}\right)^{-n}\sum_{k=0}^{n-1}\frac{(n-k-1)!}{k!}\left(\frac{-z}{4}\right)^k + (-1)^{n+1}\ln\left(\frac{z}{2}\right)I_n(z) \\ &+ \frac{1}{2}\left(\frac{-z}{2}\right)^n\sum_{k=0}^{\infty}\left(\psi(k+1)+\psi(n+k+1)\right)\frac{\left(\frac{z^2}{4}\right)^k}{k!(n+k)!}, \end{aligned}$$

where

$$\psi(1) = -\gamma \text{ (Euler's constant)} \quad \text{and} \quad \forall m \in \mathbb{N}^*,\ \psi(m+1) = \sum_{k=1}^{m}\frac{1}{k} - \gamma.$$

In particular

$$K_0(z) = -\log\left(\frac{z}{2}\right)I_0(z) + \sum_{m=0}^{\infty}\frac{\left(\frac{z}{2}\right)^{2m}}{(m!)^2}\psi(m+1), \tag{A.5}$$

so K_0 behaves like a logarithm at 0.

Decay property for the product $I_\nu K_\nu$ (see [2] and [9]) **:**

The application $(\lambda,\nu) \mapsto I_\nu(\lambda)K_\nu(\lambda)$ is strictly decreasing in each variable $(\lambda,\nu) \in (\mathbb{R}_+^*)^2$.

Beltrami's summation formula (see [41, p. 361]) **:** Let $0 < b < a$. Then

$$\forall\theta \in \mathbb{R}, \quad K_0\left(\sqrt{a^2+b^2-2ab\cos(\theta)}\right) = \sum_{m=-\infty}^{\infty} I_m(b)K_m(a)\cos(m\theta). \tag{A.6}$$

Ratio bounds (see [3]) **:**

For all $n \in \mathbb{N}$, for all $\lambda \in \mathbb{R}_+^*$, we have

$$\begin{cases} \dfrac{\lambda I_n'(\lambda)}{I_n(\lambda)} < \sqrt{\lambda^2+n^2} \\[2ex] \dfrac{\lambda K_n'(\lambda)}{K_n(\lambda)} < -\sqrt{\lambda^2+n^2} \end{cases} \tag{A.7}$$

Integral representation for the product I_nK_n (see [35, p. 140]) **:**

$$\forall n\in\mathbb{N}^*,\quad \forall\lambda>0,\quad (I_nK_n)(\lambda)=\frac{1}{2}\int_0^\infty J_0\big(2\lambda\sinh(\tfrac{t}{2})\big)e^{-nt}dt. \tag{A.8}$$

Asymptotic expension of small argument (see [1, p. 375]) **:**

$$\forall n\in\mathbb{N}^*,\quad I_n(\lambda)\underset{\lambda\to0}{\sim}\frac{\left(\frac{1}{2}\lambda\right)^n}{\Gamma(n+1)}\qquad\text{and}\qquad K_n(\lambda)\underset{\lambda\to0}{\sim}\frac{\Gamma(n)}{2\left(\frac{1}{2}\lambda\right)^n}. \tag{A.9}$$

Asymptotic expansion of high order (see [1, p. 377]) **:**

$$\forall\lambda>0,\qquad I_\nu(\lambda)\underset{\nu\to\infty}{\sim}\frac{1}{\sqrt{2\pi\nu}}\left(\frac{e\lambda}{2\nu}\right)^\nu\qquad\text{and}\qquad K_\nu(\lambda)\underset{\nu\to\infty}{\sim}\sqrt{\frac{\pi}{2\nu}}\left(\frac{e\lambda}{2\nu}\right)^{-\nu}. \tag{A.10}$$

Asymptotic expansion of high order for the product I_jK_j (see [31]) **:**

$$\forall\lambda>0,\quad \forall b\in(0,1],\quad I_n(\lambda b)K_n(\lambda)\underset{n\to\infty}{\sim}\frac{b^n}{2n}\left(\sum_{m=0}^\infty\frac{b_m(\lambda b)}{n^m}\right)\left(\sum_{m=0}^\infty(-1)^m\frac{b_m(\lambda)}{n^m}\right), \tag{A.11}$$

where for each $m\in\mathbb{N}$, $b_m(\lambda)$ is a polynomial of degree m in λ^2 defined by

$$b_0(\lambda)=1\qquad\text{and}\qquad\forall m\in\mathbb{N}^*,\ b_m(\lambda)=\sum_{k=1}^m(-1)^{m-k}\frac{S(m,k)}{k!}\left(\frac{\lambda^2}{4}\right)^k$$

and the $S(m,k)$ are Stirling numbers of second kind defined recursively by

$$\forall(m,k)\in(\mathbb{N}^*)^2,\quad S(m,k)=S(m-1,k-1)+kS(m-1,k),$$

with

$$S(0,0)=1,\quad\forall m\in\mathbb{N}^*,\ S(m,1)=1\qquad\text{and}\qquad S(m,0)=0\qquad\text{and if } m<k \text{ then } S(m,k)=0.$$

B Proof of Proposition 2.1

In this appendix, we prove the regularity result stated in Proposition 2.1. The techniques involved are now classical and the following proof follows closely the lines of the proof of [19, Prop. 4.1].

Proof. **(i)** The proof proceeds in three steps. The first step is to show the well-posedness of the function $G(\lambda,b,\cdot,\cdot,\cdot)$ from $\mathbb{R}\times B_r^{1+\alpha}\times B_r^{1+\alpha}$ to Y^α for some r small enough. Then, in the second step, we shall prove the existence and give the computation of the Gâteaux derivative of $G(\lambda,b,\cdot,\cdot,\cdot)$. Finally, in the third step, we shall prove that these Gâteaux derivatives are continuous. This will show the C^1 regularity of $G(\lambda,b,\cdot,\cdot,\cdot)$.

► **Step 1 : Show that** $G(\lambda,b,\cdot,\cdot,\cdot):\mathbb{R}\times B_r^{1+\alpha}\times B_r^{1+\alpha}\to Y^\alpha$ **is well-defined :**
For this purpose, we split G_j into two terms, the self-induced term $\mathcal{S}_j$ and the interaction term $\mathcal{I}_j$,

$$G_j(\lambda,b,\Omega,f_1,f_2)=\mathcal{S}_j(\lambda,b,\Omega,f_j)+\mathcal{I}_j(\lambda,b,f_1,f_2), \tag{B.1}$$

where

$$\mathcal{S}_j(\lambda,b,\Omega,f_j)(w):=\mathrm{Im}\left\{\left[\Omega\Phi_j(w)+(-1)^jS(\lambda,\Phi_j,\Phi_j)(w)\right]\overline{w}\overline{\Phi_j'(w)}\right\},$$
$$\mathcal{I}_j(\lambda,b,f_1,f_2):=(-1)^{j-1}\mathrm{Im}\left\{S(\lambda,\Phi_i,\Phi_j)(w)\overline{w}\overline{\Phi_j'(w)}\right\}.$$

➢ We refer to [9, Prop. 5.7] for the study of $\mathcal{S}_j$. Only the $(-1)^j$ differs, but has no consequence. We recall here the results. There exists $r\in(0,1)$ such that for all $\alpha\in(0,1)$, we have

- $\mathcal{S}_j(\lambda,b,\cdot,\cdot):\mathbb{R}\times B_r^{1+\alpha}\to Y_1^\alpha$ is of class C^1.
- The restriction $\mathcal{S}_j(\lambda,b,\cdot,\cdot):\mathbb{R}\times B_{r,\mathbf{m}}^{1+\alpha}\to Y_{\mathbf{m}}^\alpha$ is well-defined.

Moreover, we have

$$D_{f_j}\mathcal{S}_j(\lambda,b,\Omega,f_j)h_j(w)=\Omega\mathrm{Im}\left\{h_j(w)\overline{w}\overline{\Phi_j'(w)}+\Phi_j(w)\overline{w}\overline{h_j'(w)}\right\}$$
$$+(-1)^j\mathrm{Im}\left\{S(\lambda,\Phi_j,\Phi_j)(w)\overline{w}\overline{h_j'(w)}+\overline{w}\overline{\Phi_j'(w)}\big[A_1(\lambda,\Phi_j,h_j)(w)+B_1(\lambda,\Phi_j,h_j)(w)\big]\right\}, \tag{B.2}$$

where

$$A_1(\lambda,\Phi_j,h_j)(w):=\fint_{\mathbb{T}}h_j'(\tau)K_0\big(\lambda|\Phi_j(w)-\Phi_j(\tau)|\big)d\tau,$$
$$B_1(\lambda,\Phi_j,h_j)(w):=\lambda\fint_{\mathbb{T}}\Phi_j'(\tau)K_0'\big(\lambda|\Phi_j(w)-\Phi_j(\tau)|\big)\frac{\mathrm{Re}\left(\left(\overline{h_j(w)}-\overline{h_j(\tau)}\right)(\Phi_j(w)-\Phi_j(\tau))\right)}{|\Phi_j(w)-\Phi_j(\tau)|}d\tau.$$

Actually, this is the most difficult part of this proof since in this case, the integrals appearing have singular kernel and the proof uses some results about singular kernels. As we shall see in the remaining of the proof, the terms concerning $\mathcal{I}_j$ are not singular.

➢ We shall first show that for $(f_1,f_2)\in B_r^{1+\alpha}\times B_r^{1+\alpha}$, we have $\mathcal{I}_j(\lambda,b,f_1,f_2)\in C^\alpha(\mathbb{T})$. According to the algebra structure of $C^\alpha(\mathbb{T})$, it suffices to show that for $i\neq j$, $S(\lambda,\Phi_i,\Phi_j)\in C^\alpha(\mathbb{T})$. For that purpose, we consider the operator $\mathcal{T}_{ij}$ defined by

$$\forall w\in\mathbb{T},\quad \mathcal{T}_{ij}\chi(w):=\fint_{\mathbb{T}}\chi(\tau)K_0\big(\lambda|\Phi_j(w)-\Phi_i(\tau)|\big)d\tau. \tag{B.3}$$

But for $w,\tau\in\mathbb{T}$, we have taking f_1 and f_2 small functions,

$$|\Phi_1(w)-\Phi_2(\tau)|\leqslant|w-b\tau|+|f_1(w)|+|f_2(\tau)|\leqslant(1+b)+\|f_1\|_{L^\infty(\mathbb{T})}+\|f_2\|_{L^\infty(\mathbb{T})}\leqslant 2(1+b) \tag{B.4}$$

and

$$|\Phi_1(w)-\Phi_2(\tau)|\geqslant|w-b\tau|-|f_1(w)|-|f_2(\tau)|\geqslant(1-b)-\|f_1\|_{L^\infty(\mathbb{T})}-\|f_2\|_{L^\infty(\mathbb{T})}\geqslant\frac{1-b}{2}. \tag{B.5}$$

Since K_0 is continuous on $\left[\frac{\lambda(1-b)}{2},2\lambda(1+b)\right]$, we have

$$\|\mathcal{T}_{ij}\chi\|_{L^\infty(\mathbb{T})}\lesssim\|\chi\|_{L^\infty(\mathbb{T})}.$$

Moreover, taking $w_1\neq w_2\in\mathbb{T}$, we have by mean value Theorem, since from (A.4) $K_0'=-K_1$ is continuous on $\left[\frac{\lambda(1-b)}{2},2\lambda(1+b)\right]$, and left triangle inequality

$$\big|\mathcal{T}_{ij}\chi(w_1)-\mathcal{T}_{ij}\chi(w_2)\big|\lesssim\int_{\mathbb{T}}|\chi(\tau)|\left|K_0\big(\lambda|\Phi_j(w_1)-\Phi_i(\tau)|\big)-K_0\big(|\lambda||\Phi_j(w_2)-\Phi_i(\tau)|\big)\right||d\tau|$$
$$\lesssim\|\chi\|_{L^\infty(\mathbb{T})}\big|\Phi_j(w_1)-\Phi_j(w_2)\big|.$$

Using that $\Phi_j\in C^{1+\alpha}(\mathbb{T})\hookrightarrow C^\alpha(\mathbb{T})$, we conclude that

$$|\mathcal{T}_{ij}\chi(w_1)-\mathcal{T}_{ij}\chi(w_2)|\lesssim\|\chi\|_{L^\infty(\mathbb{T})}\|\Phi_j\|_{C^\alpha(\mathbb{T})}|w_1-w_2|^\alpha.$$

We deduce that

$$\|\mathcal{T}_{ij}\chi\|_{C^\alpha(\mathbb{T})}\lesssim\big(1+\|\Phi_j\|_{C^\alpha(\mathbb{T})}\big)\|\chi\|_{L^\infty(\mathbb{T})}. \tag{B.6}$$

Applying this with $\chi=\Phi_j'$, we find

$$\|S(\lambda,\Phi_i,\Phi_j)\|_{C^\alpha(\mathbb{T})}\lesssim\big(1+\|\Phi_j\|_{C^\alpha(\mathbb{T})}\big)\|\Phi_i'\|_{L^\infty(\mathbb{T})}\lesssim\big(1+\|\Phi_j\|_{C^{1+\alpha}(\mathbb{T})}\big)\|\Phi_i\|_{C^{1+\alpha}(\mathbb{T})}<\infty.$$

The last point to check is that the Fourier coefficients of $\mathcal{I}_j(\lambda,f_1,f_2)$ are real. According to the definition of the space $X^{1+\alpha}$, the mapping Φ_j has real coefficients. We deduce that the Fourier coefficients of Φ_j' are also real. Due to the stability of such property under conjugation and multiplication, we only have to prove that the Fourier coefficients of $S(\lambda,\Phi_i,\Phi_j)$ are real. This is checked by the following computations. By using (A.3) and the change of variables $\eta \mapsto -\eta$, one has

$$\begin{aligned}
\overline{S(\lambda, \Phi_i, \Phi_j)(w)} &= \overline{\fint_{\mathbb{T}} \Phi_i'(\tau) K_0\big(\lambda|\Phi_j(w) - \Phi_i(\tau)|\big) d\tau} \\
&= \overline{\frac{1}{2\mathrm{i}\pi} \int_0^{2\pi} \Phi_i'\left(e^{\mathrm{i}\eta}\right) K_0\left(\lambda|\Phi_j(w) - \Phi_i\left(e^{\mathrm{i}\eta}\right)|\right) \mathrm{i}e^{\mathrm{i}\eta} d\eta} \\
&= \frac{1}{2\pi} \int_0^{2\pi} \Phi_i'\left(e^{-\mathrm{i}\eta}\right) K_0\left(\lambda|\Phi_j(\overline{w}) - \Phi_i\left(e^{-\mathrm{i}\eta}\right)|\right) e^{-\mathrm{i}\eta} d\eta \\
&= \frac{1}{2\mathrm{i}\pi} \int_0^{2\pi} \Phi_i'\left(e^{\mathrm{i}\eta}\right) K_0\left(\lambda|\Phi_j(\overline{w}) - \Phi_i\left(e^{\mathrm{i}\eta}\right)|\right) \mathrm{i}e^{\mathrm{i}\eta} d\eta \\
&= \fint_{\mathbb{T}} \Phi_i'(\tau) K_0\big(\lambda|\Phi_j(\overline{w}) - \Phi_i(\tau)|\big) d\tau \\
&= S(\lambda, \Phi_i, \Phi_j)(\overline{w}).
\end{aligned}$$

► **Step 2 : Show the existence and compute the Gâteaux derivatives of** $G(\lambda, b, \cdot, \cdot, \cdot)$ **:**
➢ The Gâteaux derivative of $\mathcal{I}_j$ at (f_1, f_2) in the direction $h = (h_1, h_2) \in X^{1+\alpha}$ is given by

$$\begin{aligned}
D\mathcal{I}_j(\lambda, b, f_1, f_2)h &= D_{f_1}\mathcal{I}_j(\lambda, b, f_1, f_2)h_1 + D_{f_2}\mathcal{I}_j(\lambda, b, f_1, f_2)h_2 \\
&:= \lim_{t\to 0} \frac{1}{t}\big[\mathcal{I}_j(\lambda, b, f_1 + th_1, f_2) - \mathcal{I}_j(\lambda, b, f_1, f_2)\big] \\
&\quad + \lim_{t\to 0} \frac{1}{t}\big[\mathcal{I}_j(\lambda, b, f_1, f_2 + th_2) - \mathcal{I}_j(\lambda, b, f_1, f_2)\big]. \qquad (B.7)
\end{aligned}$$

The previous limits are understood in the sense of the strong topology of Y^α. As a consequence, we need to to prove first the pointwise existence of these limits and then we shall check that these limits exist in the strong topology of $C^\alpha(\mathbb{T})$. To be able to compute the Gâteaux dérivatives, we have to precise that since the beginning of this study we have identified $\mathbb{C}$ with $\mathbb{R}^2$. Hence $\mathbb{C}$ is naturally endowed with the Euclidean scalar product which writes for $z_1 = a_1 + \mathrm{i}b_1$ and $z_2 = a_2 + \mathrm{i}b_2$

$$\langle z_1, z_2\rangle := \mathrm{Re}(\overline{z_1}z_2) = \frac{1}{2}\left(\overline{z_1}z_2 + z_1\overline{z_2}\right) = a_1a_2 + b_1b_2.$$

By straightforward computations, we infer

$$\begin{aligned}
D_{f_j}\mathcal{I}_j(\lambda, b, f_1, f_2)h_j(w) = (-1)^{j-1}\mathrm{Im}\Big\{&\overline{w}\overline{h_j'(w)}S(\lambda, \Phi_i, \Phi_j)(w) \\
&+\frac{\lambda}{2}\overline{w}\overline{\Phi_j'(w)}\Big(\overline{h_j(w)}A(\lambda, \Phi_i, \Phi_j)(w) + h_j(w)B(\lambda, \Phi_i, \Phi_j)(w)\Big)\Big\}, \qquad (B.8)
\end{aligned}$$

where

$$A(\lambda, \Phi_i, \Phi_j)(w) := \fint_{\mathbb{T}} \Phi_i'(\tau) K_0'\big(\lambda|\Phi_j(w) - \Phi_i(\tau)|\big) \frac{\Phi_j(w) - \Phi_i(\tau)}{|\Phi_j(w) - \Phi_i(\tau)|} d\tau := \fint_{\mathbb{T}} \Phi_i'(\tau) K(\lambda, w, \tau) d\tau$$

and

$$B(\lambda, \Phi_i, \Phi_j)(w) := \fint_{\mathbb{T}} \Phi_i'(\tau) K_0'\big(\lambda|\Phi_j(w) - \Phi_i(\tau)|\big) \frac{\Phi_j(\overline{w}) - \Phi_i(\overline{\tau})}{|\Phi_j(w) - \Phi_i(\tau)|} d\tau = \fint_{\mathbb{T}} \Phi_i'(\tau)\overline{K(\lambda, w, \tau)} d\tau.$$

Since B differs from A only with a conjugation, then, they both satisfy the same estimates in the coming analysis. For all $w \in \mathbb{T}$, we have

$$|A(\lambda, \Phi_i, \Phi_j)(w)| \lesssim \int_{\mathbb{T}} |\Phi_i'(\tau)| K_0\big(\lambda|\Phi_j(w) - \Phi_i(\tau)|\big) |d\tau| \lesssim \|\Phi_i'\|_{L^\infty(\mathbb{T})}.$$

So

$$\|A(\lambda, \Phi_i, \Phi_j)\|_{L^\infty(\mathbb{T})} \lesssim \|\Phi_i'\|_{L^\infty(\mathbb{T})}.$$

Let $w_1 \neq w_2 \in \mathbb{T}$. let $\tau \in \mathbb{T}$. Then

$$\begin{aligned}
&|K(\lambda, w_1, \tau) - K(\lambda, w_2, \tau)| \\
&= \left| K_0'\big(\lambda|\Phi_j(w_1) - \Phi_i(\tau)|\big) \frac{\Phi_j(w_1) - \Phi_i(\tau)}{|\Phi_j(w_1) - \Phi_i(\tau)|} - K_0'\big(\lambda|\Phi_j(w_2) - \Phi_i(\tau)|\big) \frac{\Phi_j(w_2) - \Phi_i(\tau)}{|\Phi_j(w_2) - \Phi_i(\tau)|} \right| \\
&\leqslant \big|K_0'\big(\lambda|\Phi_j(w_1) - \Phi_i(\tau)|\big) - K_0'\big(\lambda|\Phi_j(w_2) - \Phi_i(\tau)|\big)\big| \\
&\quad + \big|K_0'\big(\lambda|\Phi_j(w_2) - \Phi_i(\tau)|\big)\big| \left| \frac{\Phi_j(w_1) - \Phi_i(\tau)}{|\Phi_j(w_1) - \Phi_i(\tau)|} - \frac{\Phi_j(w_2) - \Phi_i(\tau)}{|\Phi_j(w_2) - \Phi_i(\tau)|} \right|.
\end{aligned}$$

But by right and left triangle inequalities, we get

$$\begin{aligned}
&\left|\frac{\Phi_j(w_1)-\Phi_i(\tau)}{|\Phi_j(w_1)-\Phi_i(\tau)|}-\frac{\Phi_j(w_2)-\Phi_i(\tau)}{|\Phi_j(w_2)-\Phi_i(\tau)|}\right|\\
&=\left|\frac{\Phi_j(w_1)-\Phi_j(w_2)}{|\Phi_j(w_1)-\Phi_i(\tau)|}+(\Phi_j(w_2)-\Phi_i(\tau))\left(\frac{1}{|\Phi_j(w_1)-\Phi_i(\tau)|}-\frac{1}{|\Phi_j(w_2)-\Phi_i(\tau)|}\right)\right|\\
&\leqslant\frac{|\Phi_j(w_1)-\Phi_j(w_2)|}{|\Phi_j(w_1)-\Phi_i(\tau)|}+|\Phi_j(w_2)-\Phi_i(\tau)|\,\frac{||\Phi_j(w_2)-\Phi_i(\tau)|-|\Phi_j(w_1)-\Phi_i(\tau)||}{|\Phi_j(w_1)-\Phi_i(\tau)|\,|\Phi_j(w_2)-\Phi_i(\tau)|}\\
&\leqslant\frac{2\,|\Phi_j(w_1)-\Phi_j(w_2)|}{|\Phi_j(w_1)-\Phi_i(\tau)|}\\
&\lesssim|\Phi_j(w_1)-\Phi_j(w_2)|\,.
\end{aligned}$$

Hence,

$$|K(\lambda,w_1,\tau)-K(\lambda,w_2,\tau)|\lesssim|\Phi_j(w_1)-\Phi_j(w_2)|\lesssim\|\Phi_j\|_{C^\alpha(\mathbb{T})}|w_1-w_2|^\alpha.$$

Thus,

$$\|A(\lambda,\Phi_i,\Phi_j)\|_{C^\alpha(\mathbb{T})}\lesssim\|\Phi_i\|_{C^{1+\alpha}(\mathbb{T})}+\|\Phi_j\|_{C^{1+\alpha}(\mathbb{T})}.$$

We conclude that,

$$\|D_{f_j}\mathcal{I}_j(\lambda,f_1,f_2)h_j\|_{C^\alpha(\mathbb{T})}\lesssim\|h_j\|_{C^{1+\alpha}(\mathbb{T})},$$

which means that $D_{f_j}\mathcal{I}_j(\lambda,b,f_1,f_2)\in\mathcal{L}(C^{1+\alpha}(\mathbb{T}),C^\alpha(\mathbb{T}))$.

➢ Concerning the other differentiation, we have

$$\begin{aligned}
D_{f_i}\mathcal{I}_j(\lambda,b,f_1,f_2)h_i(w)&=(-1)^{j-1}\mathrm{Im}\left\{\overline{w}\overline{\Phi_j'(w)}\fint_{\mathbb{T}}h_i'(\tau)K_0\big(\lambda|\Phi_j(w)-\Phi_i(\tau)|\big)d\tau\right.\\
&-\frac{\lambda}{2}\overline{w}\overline{\Phi_j'(w)}\fint_{\mathbb{T}}h_i(\tau)\Phi_i'(\tau)K_0'\big(\lambda|\Phi_j(w)-\Phi_i(\tau)|\big)\frac{\Phi_j(\overline{w})-\Phi_i(\overline{\tau})}{|\Phi_j(w)-\Phi_i(\tau)|}d\tau\\
&\left.-\frac{\lambda}{2}\overline{w}\overline{\Phi_j'(w)}\fint_{\mathbb{T}}\overline{h_i(\tau)}\Phi_i'(\tau)K_0'\big(\lambda|\Phi_j(w)-\Phi_i(\tau)|\big)\frac{\Phi_j(w)-\Phi_i(\tau)}{|\Phi_j(w)-\Phi_i(\tau)|}d\tau\right\}\\
&:=(-1)^{j-1}\mathrm{Im}\Big\{\overline{w}\overline{\Phi_j'(w)}\big[C(\lambda,\Phi_i,\Phi_j)(h_i)(w)+D(\lambda,\Phi_i,\Phi_j)(h_i)(w)+E(\lambda,\Phi_i,\Phi_j)(h_i)(w)\big]\Big\}.
\end{aligned}\tag{B.9}$$

Using the algebra structure of $C^\alpha(\mathbb{T})$, we obtain

$$\|D_{f_i}\mathcal{I}_j(\lambda,b,f_1,f_2)h_i\|_{C^\alpha(\mathbb{T})}\lesssim\|C(\lambda,\Phi_i,\Phi_j)h_i\|_{C^\alpha(\mathbb{T})}+\|D(\lambda,\Phi_i,\Phi_j)h_i\|_{C^\alpha(\mathbb{T})}+\|E(\lambda,\Phi_i,\Phi_j)h_i\|_{C^\alpha(\mathbb{T})}.$$

From (B.6), we find

$$\|C(\lambda,\Phi_i,\Phi_j)h_i\|_{C^\alpha(\mathbb{T})}\lesssim\|h_i'\|_{L^\infty(\mathbb{T})}\leqslant\|h_i\|_{C^{1+\alpha}(\mathbb{T})}.$$

In the same way as for $A(\lambda,\Phi_i,\Phi_j)$, we infer

$$\|D(\lambda,\Phi_i,\Phi_j)h_i\|_{C^\alpha(\mathbb{T})}+\|E(\lambda,\Phi_i,\Phi_j)h_i\|_{C^\alpha(\mathbb{T})}\lesssim\|h_i\|_{L^\infty(\mathbb{T})}\leqslant\|h_i\|_{C^{1+\alpha}(\mathbb{T})}.$$

Gathering the foregoing computations leads to

$$\|D_{f_i}\mathcal{I}_j(\lambda,b,f_1,f_2)h_i\|_{C^\alpha(\mathbb{T})}\lesssim\|h_i\|_{C^{1+\alpha}(\mathbb{T})},$$

that is, $D_{f_i}\mathcal{I}_j(\lambda,b,f_1,f_2)\in\mathcal{L}(C^{1+\alpha}(\mathbb{T}),C^\alpha(\mathbb{T}))$.

➢ The last thing to check is that the convergence in (B.7) occurs in the strong topology of $C^\alpha(\mathbb{T})$. Since there are many terms involved, we shall select the more complicated one and study it. The other terms can be treated in a similar way, up to slight modifications. Let us focus on the first term of the right-hand side of (B.8). We shall prove,

$$\lim_{t\to0}S(\lambda,\Phi_i,\Phi_i+th_j)-S(\lambda,\Phi_i,\Phi_j)=0\quad\text{in}\quad C^\alpha(\mathbb{T}).$$

For more convenience, we use the following notation

$$T_{ij}(\lambda,t,w):=S(\lambda,\Phi_i,\Phi_i+th_j)(w)-S(\lambda,\Phi_i,\Phi_j)(w).$$

Consider $t>0$ such that $t\|h_j\|_{L^\infty(\mathbb{T})}<r$. According to (2.9), we get

$$\begin{aligned}
T_{ij}(\lambda,t,w)&=\fint_{\mathbb{T}}\Phi_i'(\tau)\Big[K_0\big(\lambda\,|\Phi_j(w)-\Phi_i(\tau)+th_j(w)|\,\big)-K_0\big(\lambda\,|\Phi_j(w)-\Phi_i(\tau)|\,\big)\Big]d\tau\\
&:=\fint_{\mathbb{T}}\Phi_i'(\tau)\mathbb{K}(\lambda,t,w,\tau)d\tau.
\end{aligned}$$

Applying mean value Theorem and left triangle inequality, we obtain

$$|\mathbb{K}(\lambda,t,w,\tau)|\lesssim t\|h_j\|_{L^\infty(\mathbb{T})}.$$

Consequently,

$$|T_{ij}(\lambda,t,w)|\lesssim t\|h_j\|_{L^\infty(\mathbb{T})}.$$

This implies that

$$\lim_{t\to0}\|T_{ij}(\lambda,t,\cdot)\|_{L^\infty(\mathbb{T})}=0.$$

Let us now consider $w_1\neq w_2\in\mathbb{T}$. In view of the mean value Theorem, one obtains the following estimate

$$\begin{aligned}|T_{ij}(\lambda,t,w_1)-T_{ij}(\lambda,t,w_2)|&\lesssim\int_{\mathbb{T}}|\mathbb{K}(\lambda,t,w_1,\tau)-\mathbb{K}(\lambda,t,w_2,\tau)|\,|d\tau|\\&\lesssim|w_1-w_2|\int_{\mathbb{T}}\sup_{w\in\mathbb{T}}|\partial_w\mathbb{K}(\lambda,t,w,\tau)|\,|d\tau|.\end{aligned}\tag{B.10}$$

Now remark that we can write

$$\mathbb{K}(\lambda,t,w,\tau)=\int_0^t\partial_s g(\lambda,s,w,\tau)ds\quad\text{with}\quad g(\lambda,t,w,\tau):=K_0\big(\lambda\,|\Phi_j(w)-\Phi_i(\tau)+\tau h_j(w)|\,\big).$$

According to (2.11), one obtains

$$\begin{aligned}\partial_w g(\lambda,t,w,\tau)&=\frac{\lambda}{2}K_0'\big(\lambda\,|\Phi_j(w)-\Phi_i(\tau)+th_j(w)|\,\big)\\&\times\frac{\big(\Phi_j'(w)+th_j'(w)\big)\left(\overline{\Phi_j(w)}-\overline{\Phi_i(\tau)}+t\overline{h_j(w)}\right)-\overline{w}^2\left(\overline{\Phi_j'(w)}+t\overline{h_j'(w)}\right)\left(\Phi_j(w)-\Phi_i(\tau)+th_j(w)\right)}{|\Phi_j(w)-\Phi_i(\tau)+th_j(w)|}.\end{aligned}$$

After straightforward computations, we obtain for $s\in[0,t]$,

$$|\partial_s\partial_w g(\lambda,s,w,\tau)|\lesssim1.$$

As a consequence, we infer

$$|\partial_w\mathbb{K}(\lambda,t,w,\tau)|\lesssim|t|.$$

Coming back to (B.10) and using the fact that $\alpha\in(0,1)$, we conclude

$$|T_{ij}(\lambda,t,w_1)-T_{ij}(\lambda,t,w_2)|\lesssim|t||w_1-w_2|\lesssim|t||w_1-w_2|^\alpha.$$

Therefore,

$$\lim_{t\to0}\|T_{ij}(t,\cdot)\|_{C^\alpha(\mathbb{T})}=0.$$

The second step is now achieved.

► **Step 3 : Show that the Gâteaux derivatives of $G(\lambda,b,\cdot,\cdot,\cdot)$ are continuous :**
Now we investigate for the continuity of the Gâteaux derivatives seen as operators from the neighborhood $B_r^{1+\alpha}\times B_r^{1+\alpha}$ into the Banach space $\mathcal{L}\left(X_1^{1+\alpha},Y_1^\alpha\right)$. Using the algebra structure of $C^\alpha(\mathbb{T})$, we deduce from (B.9) and (B.8) that we only have to study the continuity of the terms $S(\lambda,\Phi_i,\Phi_j)$, $A(\lambda,\Phi_i,\Phi_j)$, $B(\lambda,\Phi_i,\Phi_j)$, $C(\lambda,\Phi_i,\Phi_j)h_i$, $D(\lambda,\Phi_i,\Phi_j)h_i$ and $E(\lambda,\Phi_i,\Phi_j)h_i$. As before, we shall focus on the term $S(\lambda,\Phi_i,\Phi_j)$ for $i\neq j$ and remark that the other terms are similar. We denote

$$\Phi_1:=\mathrm{Id}+f_1,\quad\Psi_1:=\mathrm{Id}+g_1,\quad\Phi_2:=b\mathrm{Id}+f_2,\quad\Psi_2:=b\mathrm{Id}+g_2,$$

with $(f_1,f_2)\in B_r^{1+\alpha}\times B_r^{1+\alpha}$ and $(g_1,g_2)\in B_r^{1+\alpha}\times B_r^{1+\alpha}$. Let us show that

$$\|S(\lambda,\Phi_i,\Phi_j)-S(\lambda,\Psi_i,\Psi_j)\|_{C^\alpha(\mathbb{T})}\lesssim\|f_1-g_1\|_{C^{1+\alpha}(\mathbb{T})}+\|f_2-g_2\|_{C^{1+\alpha}(\mathbb{T})}.$$

According to (2.9), we get

$$\begin{aligned}S(\lambda,\Phi_i,\Phi_j)(w)-S(\lambda,\Psi_i,\Psi_j)(w)&=\fint_{\mathbb{T}}\Big[\Phi_i'(\tau)K_0\big(\lambda\,|\Phi_j(w)-\Phi_i(\tau)|\,\big)-\Psi_i'(\tau)K_0\big(\lambda\,|\Psi_j(w)-\Psi_i(\tau)|\,\big)\Big]d\tau\\&:=\fint_{\mathbb{T}}\Psi_i'(\tau)\mathbb{K}_2(\lambda,w,\tau)d\tau+\fint_{\mathbb{T}}\big(\Phi_i'(\tau)-\Psi_i'(\tau)\big)K_0\big(\lambda\,|\Phi_j(w)-\Phi_i(\tau)|\,\big)d\tau,\end{aligned}$$

where

$$\mathbb{K}_2(\lambda, w, \tau) := K_0\big(\lambda\,|\Phi_j(w) - \Phi_i(\tau)|\,\big) - K_0\big(\lambda\,|\Psi_j(w) - \Psi_i(\tau)|\,\big).$$

We have directly

$$\Big\| \fint_{\mathbb{T}} \big(\Phi_i'(\tau) - \Psi_i'(\tau)\big) K_0\big(\lambda\,|\Phi_j(\cdot) - \Phi_i(\tau)|\,\big) d\tau \Big\|_{C^\alpha(\mathbb{T})} \lesssim \|f_i' - g_i'\|_{L^\infty(\mathbb{T})} \leqslant \|f_i - g_i\|_{C^{1+\alpha}(\mathbb{T})}.$$

Now set

$$L_i(\lambda, w) := \fint_{\mathbb{T}} \mathbb{K}_2(\lambda, w, \tau)\Psi_i'(\tau) d\tau,$$

By a new use of the mean value Theorem and left triangle inequality, we obtain

$$\begin{aligned} |\mathbb{K}_2(\lambda, w, \tau)| &\lesssim \big|\, |\Phi_j(w) - \Phi_i(\tau)| - |\Psi_j(w) - \Psi_i(\tau)|\, \big| \\ &\leqslant |\Phi_j(w) - \Psi_j(w)| + |\Phi_i(\tau) - \Psi_i(\tau)| \\ &\leqslant \|\Psi_j - \Phi_j\|_{L^\infty(\mathbb{T})} + \|\Psi_i - \Phi_i\|_{L^\infty(\mathbb{T})}. \end{aligned}$$

Hence, we deduce

$$\begin{aligned} \|L_i(\lambda, \cdot)\|_{L^\infty(\mathbb{T})} &\lesssim \|\Psi_i'\|_{L^\infty(\mathbb{T})} \big(\|\Psi_j - \Phi_j\|_{L^\infty(\mathbb{T})} + \|\Psi_i - \Phi_i\|_{L^\infty(\mathbb{T})}\big) \\ &\lesssim \|f_j - g_j\|_{C^{1+\alpha}(\mathbb{T})} + \|f_i - g_i\|_{C^{1+\alpha}(\mathbb{T})}. \end{aligned}$$

Take $w_1 \neq w_2 \in \mathbb{T}$. Applying the mean value Theorem yields

$$|L_i(\lambda, w_1) - L_i(\lambda, w_2)| \lesssim |w_1 - w_2| \fint_{\mathbb{T}} \sup_{w \in \mathbb{T}} |\partial_w \mathbb{K}_2(\lambda, w, \tau)|\, |d\tau|.$$

By (2.11), we have

$$\partial_w \mathbb{K}_2(\lambda, w, \tau) = \frac{\lambda}{2}\left(\overline{\mathcal{J}(\lambda, w, \tau)} - \overline{w}^2 \mathcal{J}(\lambda, w, \tau)\right),$$

where

$$\mathcal{J}(\lambda, w, \tau) := \overline{\Phi_j'(w)}(\Phi_j(w) - \Phi_i(\tau))K_0'\left(\lambda|\Phi_j(w) - \Phi_i(\tau)|\right) - \overline{\Psi_j'(w)}(\Psi_j(w) - \Psi_i(\tau))K_0'\left(\lambda|\Psi_j(w) - \Psi_i(\tau)|\right).$$

Notice that it can be written in the following form

$$\mathcal{J}(\lambda, w, \tau) = \mathcal{J}_1(\lambda, w, \tau) + \mathcal{J}_2(\lambda, w, \tau) + \mathcal{J}_3(\lambda, w, \tau),$$

with

$$\begin{aligned} \mathcal{J}_1(\lambda, w, \tau) &:= \overline{\Phi_j'(w)}\left[(\Phi_j - \Psi_j)(w) - (\Phi_i - \Psi_i)(\tau)\right] K_0'\left(\lambda|\Phi_j(w) - \Phi_i(\tau)|\right), \\ \mathcal{J}_2(\lambda, w, \tau) &:= \left[\overline{\Phi_j'(w)} - \overline{\Psi_j'(w)}\right]\left[\Psi_j(w) - \Psi_i(\tau)\right] K_0'\left(\lambda|\Psi_j(w) - \Psi_i(\tau)|\right), \\ \mathcal{J}_3(\lambda, w, \tau) &:= \overline{\Phi_j'(w)}\left[\Psi_j(w) - \Psi_i(\tau)\right]\left[K_0'\left(\lambda|\Phi_j(w) - \Phi_i(\tau)|\right) - K_0'\left(\lambda|\Psi_j(w) - \Psi_i(\tau)|\right)\right]. \end{aligned}$$

By the same techniques as already used above, we get

$$\|\partial_w \mathbb{K}_2(\lambda, \cdot, \tau)\|_{L^\infty(\mathbb{T})} \lesssim \|f_j - g_j\|_{C^{1+\alpha}(\mathbb{T})} + \|f_i - g_i\|_{C^{1+\alpha}(\mathbb{T})}.$$

We deduce that

$$\|S(\lambda, \Phi_i, \Phi_j) - S(\lambda, \Psi_i, \Psi_j)\|_{C^\alpha(\mathbb{T})} \lesssim \|f_j - g_j\|_{C^{1+\alpha}(\mathbb{T})} + \|f_i - g_i\|_{C^{1+\alpha}(\mathbb{T})}.$$

(ii) Looking at Proposition 2.1, it is sufficient to prove the preservation of the **m**-fold symmetry. Let r be as in Proposition 2.1. Let $(f_1, f_2) \in B^{1+\alpha}_{r,\mathbf{m}} \times B^{1+\alpha}_{r,\mathbf{m}}$. Let Φ_1 and Φ_2 be the associated conformal maps

$$\Phi_1(z) = z + \sum_{n=0}^{\infty} \frac{a_n}{z^{\mathbf{m}n-1}} \qquad \text{and} \qquad \Phi_2(z) = bz + \sum_{n=0}^{\infty} \frac{b_n}{z^{\mathbf{m}n-1}}.$$

One easily obtains

$$\forall j \in \{1, 2\}, \quad \forall w \in \mathbb{T}, \quad \Phi_j\left(e^{\frac{2i\pi}{\mathbf{m}}} w\right) = e^{\frac{2i\pi}{\mathbf{m}}} \Phi_j(w) \quad \text{and} \quad \Phi_j'\left(e^{\frac{2i\pi}{\mathbf{m}}} w\right) = \Phi_j'(w).$$

Hence, by using the change of variables $\tau \mapsto e^{\frac{2i\pi}{\mathbf{m}}}\tau$, we have for all $(i,j) \in \{1,2\}^2$ and for all $w \in \mathbb{T}$,

$$\begin{aligned}
S(\lambda, \Phi_i, \Phi_j)\left(e^{\frac{2i\pi}{\mathbf{m}}}w\right) &= \fint_{\mathbb{T}} \Phi_i'(\tau) K_0\left(\lambda \left|\Phi_j\left(e^{\frac{2i\pi}{\mathbf{m}}}w\right) - \Phi_i(\tau)\right|\right) d\tau \\
&= e^{\frac{2i\pi}{\mathbf{m}}} \fint_{\mathbb{T}} \Phi_i'\left(e^{\frac{2i\pi}{\mathbf{m}}}\tau\right) K_0\left(\lambda \left|\Phi_j\left(e^{\frac{2i\pi}{\mathbf{m}}}w\right) - \Phi_i\left(e^{\frac{2i\pi}{\mathbf{m}}}\tau\right)\right|\right) d\tau \\
&= e^{\frac{2i\pi}{\mathbf{m}}} \fint_{\mathbb{T}} \Phi_i'(\tau) K_0\left(\lambda \left|\Phi_j(w) - \Phi_i(\tau)\right|\right) d\tau \\
&= e^{\frac{2i\pi}{\mathbf{m}}} S(\lambda, \Phi_i, \Phi_j)(w).
\end{aligned}$$

By definition (2.8) of G_j, this immediately implies that

$$\forall j \in \{1,2\}, \quad \forall w \in \mathbb{T}, \quad G_j(\lambda, b, \Omega, f_1, f_2)\left(e^{\frac{2i\pi}{\mathbf{m}}}w\right) = G_j(\lambda, b, \Omega, f_1, f_2)(w).$$

So

$$G(\lambda, b, \cdot, \cdot, \cdot) : \mathbb{R} \times B^{1+\alpha}_{r,\mathbf{m}} \times B^{1+\alpha}_{r,\mathbf{m}} \to Y^{\alpha}_{\mathbf{m}}.$$

(iii) Fix $j \in \{1,2\}$. By (B.1) and (B.2), we have for $f_j \in B^{1+\alpha}_r$ and $h_j \in C^{1+\alpha}(\mathbb{T})$,

$$\begin{aligned}
\partial_\Omega D_{f_j} G_j(\lambda, b, \Omega, f_j)(h_j)(w) &= \partial_\Omega D_{f_j} \mathcal{S}_j(\lambda, b, \Omega, f_j)(h_j)(w) \\
&= \operatorname{Im}\left\{h_j(w)\overline{w}\overline{\Phi_j'(w)} + \Phi_j(w)\overline{w}\overline{h_j'(w)}\right\}.
\end{aligned}$$

As a consequence, we deduce that for $(f_j, g_j) \in (B^{1+\alpha}_r)^2$ and $h_j \in C^{1+\alpha}(\mathbb{T})$,

$$\left\|\partial_\Omega D_{f_j} G_j(\lambda, b, \Omega, f_j)(h_j) - \partial_\Omega D_{f_j} G_j(\lambda, b, \Omega, g_j)(h_j)\right\|_{C^\alpha(\mathbb{T})} \lesssim \|f_j - g_j\|_{C^{1+\alpha}(\mathbb{T})} \|h_j\|_{C^{1+\alpha}(\mathbb{T})}.$$

This proves the continuity of $\partial_\Omega DG(\lambda, b, \cdot, \cdot, \cdot) : \mathbb{R} \times B^{1+\alpha}_r \times B^{1+\alpha}_r \to \mathcal{L}(X^{1+\alpha}, Y^{\alpha})$ and achieves the proof of Proposition 2.1. □

C Crandall-Rabinowitz's Theorem

Now, we recall the classical Crandall-Rabinowitz's Theorem. This result was first proved in [7] and it is one of the most common theorems appearing in the bifurcation theory. A convenient reference in the subject is [32]. We briefly explain the core of local bifurcation theory.

Consider a function $F : (\Omega, x) \in \mathbb{R} \times X \mapsto F(\Omega, x) \in Y$ with X and Y two Banach spaces. Assume that for all Ω in a non-empty interval I we have $F(\Omega, 0) = 0$. This provides a line of solutions

$$\big\{(\Omega, 0), \quad \Omega \in I\big\}.$$

Now take some $(\Omega_0, 0)$ with $\Omega_0 \in I$. The implicit function Theorem explains that if $D_x F(\Omega_0, 0)$ is invertible, then the line $\{(\Omega, 0), |\Omega - \Omega_0| \leqslant \varepsilon\}$ is the only curve of solutions close to $(\Omega_0, 0)$, i.e. for ε small enough. (Local) bifurcation theory is the study of situations where this is not true, that is, close to $(\Omega_0, 0)$ there exists (at least) another line of solutions. In this case, we say that $(\Omega_0, 0)$ is a bifurcation point. Crandall-Rabinowitz's Theorem gives sufficient conditions to construct a bifurcation curve and states as follows.

Theorem C.1 (Crandall-Rabinowitz)**.** *Let X and Y be two banach spaces. Let V be a neighborhood of 0 in X and let*

$$\begin{array}{rccc} F: & \mathbb{R} \times V & \to & Y \\ & (\Omega, x) & \mapsto & F(\Omega, x) \end{array}$$

be a function with the following properties

(i) (Trivial solution) $\forall\, \Omega \in \mathbb{R}, F(\Omega, 0) = 0$.

(ii) (Regularity) $\partial_\Omega F$, $D_x F$ and $\partial_\Omega D_x F$ exist and are continuous.

(iii) (Fredholm property) $\ker(D_x F(0,0)) = \langle x_0 \rangle$ *and $Y/R(D_x F(0,0))$ are one dimensional and $R(D_x F(0,0))$ is closed in Y.*

(iv) (Transversality assumption) $\partial_\Omega D_x F(0,0) x_0 \notin R(D_x F(0,0))$.

If χ is any complement of $\ker(D_xF(0,0))$ *in X, then there exist a neighborhood U of $(0,0)$, an interval $(-a,a)$ $(a>0)$ and continuous functions*

$$\psi:(-a,a)\to\mathbb{R}\quad\text{and}\quad\phi:(-a,a)\to\chi$$

such that $\psi(0)=0$, $\phi(0)=0$ and

$$\Big\{(\Omega,x)\in U\quad\text{s.t.}\quad F(\Omega,x)=0\Big\}=\Big\{\big(\psi(s),sx_0+s\phi(s)\big)\quad\text{s.t.}\quad|s|<a\Big\}\cup\Big\{(\Omega,0)\in U\Big\}.$$

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